Integration of Brownian vector fields

Yves Le Jan and Olivier Raimond

Université Paris-Sud

Mathématiques

Bâtiment 425

91405 Orsay cedex

e-mail: Yves.LeJan@math.u-psud.fr; Olivier.Raimond@math.u-psud.fr

Summary. Using the Wiener chaos decomposition, we show that strong solutions of non Lipschitzian S.D.E.'s are given by random Markovian kernels. The example of Sobolev flows is studied in some detail, exhibiting interesting phase transitions.

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Introduction.

The purpose of this paper is to present an extended notion of strong solution to S.D.E.'s driven by Wiener processes. These solutions can be defined on rather general spaces, in the context of Dirichlet forms.

More interestingly, they are not always given by flows of maps but by flows of Markovian kernels, which means splitting can occur. Coalescent flows also appear as solutions of these S.D.E.'s. Conditions are given under which coalescence and splitting occur or not.

A variety of examples are studied. The case of isotropic Sobolev flows on the sphere or on the Euclidean space shows in particular that splitting is related to hyperinstability and coalescence to hyperstability. These notions (which will be developped in sections 9 and 10) are related to the explosion of the Lyapunov exponent toward $+\infty$ and $-\infty$.

The typical example we have in mind is the Brownian motion on a Riemannian manifold. We consider a covariance on vector fields which induces the Riemannian metric on each tangent space. When the covariance function has enough regularity, it is known that one can solve the linear S.D.E. driven by the canonical Wiener process associated to this covariance (or in other terms to the local characteristics associated to this covariance (see section 3 below)) and get a multiplicative Brownian motion on the diffeomorphism group, which moves every point as a Brownian motion (see Le Jan and Watanabe [23] or Kunita [18]). But models related to turbulence theory produce natural examples where the regularity condition is not satisfied. Except for the work of Darling [7], where strong solutions are not considered, these S.D.E.'s have not been really studied. The idea is to define the solutions by their Wiener chaos expansion in terms of the heat semigroup. We call it the statistical solution. A similar expansion was given by Krylov and Veretennikov in [17], for S.D.E.'s with strong solutions.

In this form, they appear as a semigroup of operators, and the fact that these operators are Markovian is not clearly visible in the formula. To prove this, we consider an independent realization of the Brownian motion on the manifold and couple it with the given Wiener process on vector fields using certain martingales. Then, the Markovian random operators which constitute the strong solution are obtained by filtering the Brownian motion with respect to this Wiener process. They determine the law of a canonical weak solution of the equation

given the Wiener process on vector fields. This construction has been adequately generalized to be presented in the case of symmetric diffusions on a locally compact metric space. It is a convenient and well studied framework but this assumption could clearly be relaxed (in particular to the framework of coercive forms). Relations with particle representations and filtering of S.P.D.E.'s can be observed (see Kurtz and Xiong [19]).

The example of Sobolev flows is studied in details on Euclidean spaces and spheres and is of major interest especially in dimension 2 and 3 where an interesting phase diagram is given in terms of the two parameters determining the Sobolev norm on vector fields : The differentiability index and the relative weight of gradients and divergence free fields (compressibility).

Some of these results have been given in the note [21] and a preliminary version of this work was released in [22]. They are directly connected and partially motivated by a series of works of Gawedzki, Kupiainen and al on turbulent advection ([2], [13] and [14])

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1 Covariance function on a manifold.

Let X be a manifold. A covariance function C on T^*X is a map from T^*X^2 in $\mathbb{R}$ such that, for any $(x,y) \in X^2$, C restricted to $T^*_xX \times T^*_yX$ is bilinear and such that for any n-uples $(\xi_1, ..., \xi_n)$ of T^*X,

$$\sum_{i,j} C(\xi_i, \xi_j) \geq 0. \tag{1.1}$$

For any $\xi = (x,u) \in T^*X$, let C_ξ be the vector field such that for any $\xi' = (y,v) \in T^*X$,

$$\langle C_\xi(y), v\rangle = C(\xi, \xi').$$

Let H_0 be the vector space generated by the vector fields C_ξ. Let us define the bilinear form on H_0, $\langle ., .\rangle_H$ such that

$$\langle C_\xi, C_{\xi'}\rangle_H = C(\xi, \xi'). \tag{1.2}$$

As equation (1.1) is satisfied, the bilinear form $\langle ., .\rangle_H$ is a scalar product on H_0. We denote $\|.\|_H$ the norm associated to $\langle ., .\rangle_H$.

Let H be the separate completion of H_0 with respect to $\|.\|_H$. $(H, \langle ., .\rangle_H)$ is a separable Hilbert space and we will design it as the self–reproducing space associated to the covariance function C. H is constituted of vector fields on X and for any $h \in H$ and any $\xi = (x, u) \in T^*X$,

$$\langle C_\xi, h\rangle_H = \langle h(x), u\rangle. \tag{1.3}$$

Let $(e_k)_k$ be an orthonormal basis of H, then equation (1.3) implies that for any $\xi = (x, u) \in T^*X$,

$$C_\xi = \sum_k \langle e_k(x), u\rangle e_k. \tag{1.4}$$

This equation implies that for any $\xi' = (y, v) \in T^*X$,

$$C(\xi, \xi') = \sum_k \langle e_k(x), u\rangle\langle e_k(y), v\rangle. \tag{1.5}$$

Therefore

$$C = \sum_k e_k \otimes e_k. \tag{1.6}$$

Remark 1.1 *On the other hand, if we start with a countable family of vector fields $(V_k)_k$, such that for any $\xi = (x, u) \in T^*X$, $\sum_k \langle V_k(x), u\rangle^2 < \infty$, it is possible to define a covariance function on X by the formula*

$$C = \sum_k V_k \otimes V_k.$$

Examples of isotropic covariances are given in section 9 and 10. See also [1].

Assume now a Riemannian metric $\langle ., .\rangle_x$ is given on X, the linear bundles TX and T^*X can be identified. We will now suppose that the covariance is *bounded by the metric* i.e that

$$C(\xi, \xi) \leq \langle u, u\rangle_x$$

for any $\xi = (x, u) \in T^*X$. Note that this condition implies that $|h(x)|_x \leq \|h\|_H$ for any $h \in H$.

Let us denote by $m(dx)$ the volume element on X. Given any differentiable function f such that $|\nabla f|$ is square integrable, we can map it linearly into Df in the Hilbert tensor product $L^2(m)\hat{\otimes}H$ setting $\langle Df, g \otimes h\rangle = \int_X g(x)\langle \nabla f(x), h(x)\rangle_x \, m(dx)$ for all $g \in L^2(m)$ and $h \in H$.

Note that

$$\|Df\|_H^2(x) \le |\nabla f(x)|^2 \tag{1.7}$$

(This notation comes from the identification $L^2(m)\hat{\otimes}H$ with the L^2 space of H valued functions on X) and that

$$\|Df\|^2_{L^2(m)\otimes H} \le \int |\nabla f|^2 dm. \tag{1.8}$$

2 Covariance function bounded by a Dirichlet form.

We can extend these notions to the framework of local Dirichlet forms. Let X be a locally compact separable metric space and m be a positive Radon measure on X such that $\mathrm{Supp}[m] = X$.

Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet space, $\mathcal{F} \subset L^2(X, m)$. We will suppose that the Dirichlet form is local and conservative. We will denote P_t, the associated Markovian semigroup, A its generator and $\mathcal{D}(A)$ the domain of A. We will also suppose that m is an invariant measure, hence that $P_t 1 = 1$. We will also assume that for any $f \in \mathcal{F}_b = L^\infty(m) \cap \mathcal{F}$, there exists $\Gamma(f,f) \in L^1(m)$ such that for any $g \in \mathcal{F}_b$,

$$2\mathcal{E}(fg, f) - \mathcal{E}(f^2, g) = \int g\Gamma(f,f)\ dm. \tag{2.1}$$

Γ can be extended to $\mathcal{F}$ and we denote $\Gamma(f,g)$ the $L^1(m)$-valued bilinear form on $\mathcal{F}^2$, where for any $(f,g) \in \mathcal{F}^2$, $\Gamma(f,g) = \frac{1}{4}\left(\Gamma(f+g, f+g) - \Gamma(f-g, f-g)\right)$. A sufficient condition for the existence of Γ (see corollary 4.2.3 in [3]) is that $\mathcal{D}(A)$ contains a subspace E of $\mathcal{D}(A)$, dense in $\mathcal{F}$, such that

$$\forall f \in E, \qquad f^2 \in \mathcal{D}(A).$$

Then, for $(f,g) \in E^2$,

$$\Gamma(f,g) = A(fg) - f\ Ag - g\ Af. \tag{2.2}$$

A necessary and sufficient condition for the existence of the energy density (or carré du champ operator) Γ is given in theorem 4.2.2 in [3].

Fundamental example 2.1 *X is a Riemannian manifold with the metric $\langle .,. \rangle$, m is the volume measure, $\mathcal{F} = H^1(X)$ and for any $(f,g) \in \mathcal{F}^2$,*

$$\mathcal{E}(f,g) = \frac{1}{2} \int_X \langle \nabla f, \nabla g \rangle \; dm.$$

In this case, $\Gamma(f,g) = \langle \nabla f, \nabla g \rangle$.

Let H be a separable Hilbert space and D a linear map from $\mathcal{F}$ into the Hilbert tensor product $L^2(m)\hat{\otimes}H$ such that, for any $f \in \mathcal{F}$

$$\|Df(x)\|_H^2 \leq \Gamma(f,f)(x) \tag{2.3}$$

$m(dx)$–a.e. The most interesting case is when there is equality in equation (2.3).

We define the covariance function C as a bilinear map from $\mathcal{F} \times \mathcal{F}$ into $L^2(m \otimes m)$ by

$$\langle C(f,g), u \otimes v \rangle_{L^2(m\otimes m)} = \int_{X^2} \langle Df(x), Dg(y) \rangle_H u(x)v(y) \; m(dx)m(dy). \tag{2.4}$$

Note that

$$\langle C(f,f), u \otimes u \rangle_{L^2(m\otimes m)} \leq 2\mathcal{E}(f,f) \|u\|_{L^2(m)}^2. \tag{2.5}$$

We will say that C is a covariance function bounded by the Dirichlet form $(\mathcal{E}, \mathcal{F})$.

Remark 2.2 *Alternatively, we could define the covariance as a positive bilinear map from $\mathcal{F} \times \mathcal{F}$ in $L^2(m \otimes m)$ (i.e such that for any $u_i \in L^2(m)$ and any $f_i \in \mathcal{F}$,*

$$\int \sum_{i,j} u_i \otimes u_j \; C(f_i, f_j) \; dm^{\otimes 2} \geq 0.) \tag{2.6}$$

such that

$$\langle C(f,f), u \otimes u \rangle_{L^2(m\otimes m)} \leq 2\mathcal{E}(f,f) \|u\|_{L^2(m)}^2 \tag{2.7}$$

and construct as before a Hilbert space such that (2.3) holds.

Indeed, C induces a linear map $\tilde{C}$ from $L^2(m) \otimes \mathcal{F}$ into $L^2(m)\hat{\otimes}\mathcal{F}$ such that

$$\langle \tilde{C}(u \otimes f), v \otimes g \rangle_{L^2(m)\otimes\mathcal{F}} = \langle C(f,g), u \otimes v \rangle_{L^2(m\otimes m)}.$$

We define H as the separable closure of the space H_0 spanned by elements of the form $\tilde{C}(u \otimes f)$, with $u \otimes f \in L^2(m) \otimes \mathcal{F}$, and equipped with the scalar product

$$\langle \tilde{C}(u \otimes f), \tilde{C}(v \otimes g) \rangle_H = \langle C(f,g), u \otimes v \rangle_{L^2(m\otimes m)}.$$

And for $f \in \mathcal{F}$, Df is defined such that for any $u \otimes v \otimes g \in L^2(m) \otimes L^2(m) \otimes \mathcal{F}$,

$$\langle Df, u \otimes \tilde{C}(v \otimes g) >_{L^2(m)\hat{\otimes} H} = \langle C(f,g), u \otimes v \rangle_{L^2(m\otimes m)}.$$

For any $h \in H$ and $f \in \mathcal{F}$ define $D_h f = \langle Df, h \rangle_H$ which belongs to $L^2(m)$. Then for any orthonormal basis $(e_k)_k$ of H,

$$C = \sum_k D_{e_k} \otimes D_{e_k}. \tag{2.8}$$

Moreover, for any $f \in \mathcal{F}$,

$$\|Df\|_H^2 = \sum_k (D_{e_k} f)^2. \tag{2.9}$$

Remark that condition (2.3) implies that for any finite family $(u_i, f_i) \in L^\infty(m) \times \mathcal{F}$,

$$\sum_{i,j} u_i u_j D(f_i, f_j) \leq \sum_{i,j} u_i u_j \Gamma(f_i, f_j), \tag{2.10}$$

where $D(f,g)$ denotes $\langle Df(x), Dg(x) \rangle_H = \sum_k D_{e_k} f(x) D_{e_k} g(x)$. When the u_i are step functions with discontinuities in a set of zero measure, (2.10) is satisfied as $\sum_{i,j} u_i u_j D(f_i, f_j) = |D(\sum_i u_i f_i)|^2$. Then, we can extend to any family (u_i) by density in $L^2(\Gamma(f,f)dm)$ for every $f \in \mathcal{F}$.

Remark 2.3 *It is clear that given a covariance C on T^*X as in Section 1, we can build the self-reproducing space H consisting of vector fields and the mapping $D : H^1(X) \to L^2(m)\hat{\otimes} H$ so as to construct a covariance function as in Section 2. Now suppose conversely that we have a separable Hilbert space H, a linear map D and a covariance C as in Section 2, and suppose we are in the Riemannian case. The condition $\|Df(x)\|_H^2 \leq \Gamma(f,f)(x) = |\nabla f(x)|^2$ implies that $C(f,g)(x,y)$ depends only on $\nabla f(x)$ and $\nabla g(y)$, and so there is a covariance $\tilde{C}$ say on T^*X so that $C(f,g)(x,y) = \tilde{C}(\nabla f(x), \nabla g(y))$. So in the Riemannian case, any Section 2 covariance function reduces to a Section 1 covariance function.*

Further, we can now assume without any loss of generality that the separable Hilbert space H is the self-reproducing space corresponding to $\tilde{C}$ and thus consists of vector fields.

Remark 2.4 *The bilinear mapping D is a derivation : for any $h \in H$ and any $f \in \mathcal{F}$ such that $f^2 \in \mathcal{F}$,*

$$D_h f^2 = 2 f D_h f. \tag{2.11}$$

Note that in the Riemanian manifold case (fundamental example 2.1), $D_h f = \nabla_h f$ when $\Gamma = D$.

Proof. We first make the remark that

$$\sum_k (D_{e_k} f^2 - 2f D_{e_k} f)^2 = D(f^2, f^2) - 4f D(f^2, f) + 4f^2 D(f, f).$$

Integrating this relation with respect to m and using (2.10), we get that

$$\int \sum_k (D_{e_k} f^2 - 2f D_{e_k} f)^2 \; dm \leq \int \left(\Gamma(f^2, f^2) - 4f\Gamma(f^2, f) + 4f^2 \Gamma(f, f) \right) \; dm = 0.$$

This implies that for every k, $D_{e_k} f^2 - 2f D_{e_k} f = 0$. □

3 Construction of the statistical solutions.

In the fundamental example 2.1, when X is a Riemannian manifold, C is smooth and when equality holds in (2.3), it is well known (see [23] and [18]) that a stochastic flow of diffeomorphisms on X can be associated with C. Then, with the notations of definition 2.1 in [23], the local characteristics of the flow are (A, L) where $A = C$ and L is the Laplacian on X.

The object of this section is to show that in the general situation considered above, it is always possible to define a flow of Markovian kernels associated with C and $(\mathcal{E}, \mathcal{F})$ (which is induced by the stochastic flow when C is smooth).

Let be given a covariance function C bounded by a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on a locally compact separable metric space as in the preceding section (equation (2.3) is satisfied). Let W_t be a cylindrical Brownian motion on H defined on some probability space $(\Omega, \mathcal{A}, P)$, i.e a Gaussian process indexed by $H \times \mathbb{R}^+$ with covariance matrix $\mathrm{cov}(W_t(h), W_s(h')) = s \wedge t \; \langle h, h' \rangle_H$. Set $W_t^k = W_t(e_k)$. $(W_t^k; \; k \in \mathbb{N})$ is a sequence of independent Wiener processes and we can represent W_t by $\sum_k W_t^k e_k$. Informally, the law of W_t is given by

$$\frac{1}{Z} e^{-\frac{1}{2} \int_0^\infty \|\dot{W}_t\|_H^2 \; dt} DW.$$

Let $\mathcal{F}_t = \sigma(W_s^k; \; k \in \mathbb{N}; s \leq t) = \sigma(W_s; \; s \leq t)$.

Proposition 3.1 *Let $S_t^0 = P_t$. We can define a sequence S_t^n of random operators on $L^2(m)$ such that $E[(S_t^n f)^2] \le P_t f^2$ in $L^1(m)$ and S_t^n is $\mathcal{F}_t$-measurable, by the recurrence formula, in $L^2(m \otimes P)$ (i.e in the Hilbert tensor product $L^2(m)\hat{\otimes}L^2(P)$)*

$$S_t^{n+1} f = P_t f + \sum_k \int_0^t S_s^n(D_{e_k} P_{t-s} f))\ dW_s^k. \tag{3.1}$$

Remark. The stochasic integral in equation (3.1) here makes sense as a Hilbert valued Itô integral. Recall that given a real Wiener process W_t and a Hilbert space H, for any F progressively measurable in $L^2(P_W \otimes dt)\hat{\otimes}H$ and any $h \in H$, $\langle \int_s^t F(u) dW_u, h\rangle_H = \int_s^t \langle F(u), h\rangle_H dW_u$ and $E[\| \int_s^t F(u) dW_u\|_H^2] = \int_s^t \|F(u)\|_H^2 du$.

Proof. Suppose we are given S_t^n, a $\mathcal{F}_t$-measurable random operator on $L^2(m)$ such that $E[(S_t^n f)^2] \le P_t f^2$.

Let $f \in L^2(m)$. For any positive t, $P_t f \in \mathcal{F}$ and $D_{e_k} P_{t-s} f$ is well defined and belongs to $L^2(m)$.

$$\begin{aligned}
E[(S_t^{n+1} f)^2] &= (P_t f)^2 + \sum_k \int_0^t E\left[(S_s^n(D_{e_k} P_{t-s} f))^2\right]\ ds \quad m-a.e. \\
&\le (P_t f)^2 + \int_0^t P_s(|DP_{t-s} f|^2)\ ds \\
&\le (P_t f)^2 + \int_0^t P_s(\Gamma(P_{t-s} f, P_{t-s} f))\ ds.
\end{aligned}$$

For $f \in L^\infty(m) \cap L^2(m)$, $\frac{\partial}{\partial s} P_s((P_{t-s} f)^2) = P_s(\Gamma(P_{t-s} f, P_{t-s} f))$ and

$$P_t f^2 = (P_t f)^2 + \int_0^t P_s(\Gamma(P_{t-s} f, P_{t-s} f))\ ds. \tag{3.2}$$

An approximation by truncation shows that equation (3.2) remains true for $f \in L^2(m)$ and $E[(S_t^{n+1} f)^2] \le P_t f^2$. □

Remark. *The definition of S_t^n is independent of the choice of the basis on H.*

In the following, we will use the canonical realization of the processes W_t^k. They will be defined as coordinate functions on $\Omega = C(\mathbb{R}^+, \mathbb{R})^{\mathbb{N}}$, with the product Wiener measure P. We note θ_t the natural shift on Ω, such that $W_{t+s}^k - W_t^k = W_s^k \circ \theta_t$.

Recall that an operator on $L^2(m)$ is called Markovian if and only if it preserves positivity and maps 1 into 1 (or more precisely, if m is not finite, if its natural extension to positive functions maps 1 into 1).

Theorem 3.2 *The family of random operators S_t^n converges in $L^2(P)$ towards a one parameter family of $\mathcal{F}_t$–adapted Markovian operators S_t such that*

a) $S_{t+s} = S_t(S_s \circ \theta_t)$, for any $s, t \geq 0$;

b) $\forall f \in L^2(m)$, $S_t f$ is uniformly continuous with respect to t in $L^2(m \otimes P)$;

c) $E[(S_t f)^2] \leq P_t f^2$, for any $f \in L^2(m)$;

d) $S_t f = P_t f + \sum_k \int_0^t S_s(D_{e_k} P_{t-s} f)\ dW_s^k$, for any $f \in L^2(m)$;

e) $S_t f = f + \sum_k \int_0^t S_s(D_{e_k} f)\ dW_s^k + \int_0^t S_s(Af)\ ds$, for any $f \in \mathcal{D}(A)$.

S_t is uniquely characterized by c) and d) or by a), c) and e). When $\Gamma = D$, we call it the statistical solution of the S.D.E. (see 3.22 below)

$$\forall f \in \mathcal{D}(A): \quad df(X_t) = \sum_k D_{e_k} f(X_t)\ dW_t^k + Af(X_t)\ dt. \tag{3.3}$$

Note that this S.D.E. does not always have a strong solution in the usual sense.

Proof. The convergence of S_t^n is immediate since for any $n \geq 1$, $J_t^n f = S_t^n f - S_t^{n-1} f$ is in the Hilbert tensor product of the n-th Wiener chaos of $L^2(P)$ with $L^2(m)$, $S_t f = P_t f + \sum_{n=1}^{\infty} J_t^n f$ and $(P_t f)^2 + \sum_{n \geq 1} E[(J_t^n f)^2] = \lim_{n \to \infty} E[(S_t^n f)^2] \leq P_t f^2$. It is clear that S_t is $\mathcal{F}_t$–adapted and satisfies c). d) is obtained taking the limit in the recurrence formula of the proposition.

Since

$$J_t^n f = \sum_{k_1, \ldots, k_n} \int_{0 < s_1 < \ldots < s_n < t} P_{s_1} D_{e_{k_1}} P_{s_2 - s_1} \ldots D_{e_{k_n}} P_{t - s_n} f\ dW_{s_1}^{k_1} \ldots dW_{s_n}^{k_n}$$

we have $J_{t+s}^n = \sum_{k \leq n} J_t^k (J_s^{n-k} \circ \theta_t)$ (the k-th term corresponds to

$$\sum_{k_1, \ldots, k_n} \int_{0 < s_1 < \ldots < s_k < s < s_{k+1} < \ldots < s_n < t+s} P_{s_1} D_{e_{k_1}} P_{s_2 - s_1} \ldots D_{e_{k_n}} P_{t+s-s_n} f\ dW_{s_1}^{k_1} \ldots dW_{s_n}^{k_n}).$$

We deduce a) from this relation.

The uniqueness of a solution of d) verifying c) follows directly from the uniqueness of the Wiener chaos decomposition, obtained by iteration of d) : Let T_t design another solution of d) and c) then for any $f \in L^2(m)$ and any integer n,

$$T_t f = S_t^{n-1} f + \sum_{k_1, \ldots, k_n} \int_{0 < s_1 < \ldots < s_n < t} T_{s_1} D_{e_{k_1}} P_{s_2 - s_1} \ldots D_{e_{k_n}} P_{t - s_n} f\ dW_{s_1}^{k_1} \ldots dW_{s_n}^{k_n}.$$

The second term of the right hand side of the preceding equation is orthogonal to the first one since its integrands are L^2. Indeed :

$$\begin{aligned}
\sum_{k_1,\ldots,k_n} E\left[\int_{0<s_1<\ldots<s_n<t} \left(T_{s_1} D_{e_{k_1}} P_{s_2-s_1} \ldots D_{e_{k_n}} P_{t-s_n} f\right)^2 \, ds_1 \ldots ds_n\right] \;\leq& \\
\leq \quad & \sum_{k_1,\ldots,k_n} \int_{0<s_1<\ldots<s_n<t} P_{s_1}(|D_{e_{k_1}} P_{s_2-s_1} \ldots D_{e_{k_n}} P_{t-s_n} f|^2) \, ds_1 \ldots ds_n \\
\leq \quad & \sum_{k_2,\ldots,k_n} \int_{0<s_2<\ldots<s_n<t} P_{s_2}(|D_{e_{k_2}} P_{s_3-s_2} \ldots D_{e_{k_n}} P_{t-s_n} f|^2) \, ds_2 \ldots ds_n
\end{aligned}$$

using equation (2.3) and (3.2) and by induction is smaller than $P_t f^2$.

This proves that the Wiener chaos decomposition of $T_t f$ and $S_t f$ are the same and therefore $T_t = S_t$.

Proof of b). Let us remark that for any positive ε, $S_{t+\varepsilon} - S_t = S_t(S_\varepsilon \circ \theta_t - I)$. As S_t and $S_\varepsilon \circ \theta_t$ are independent and m is invariant under P_t , for any $f \in L^2(m)$

$$\begin{aligned}
\int E[(S_{t+\varepsilon} f - S_t f)^2] \, dm \;&\leq\; \int E[P_t(S_\varepsilon \circ \theta_t f - f)^2] \, dm \\
&\leq\; \int E[(S_\varepsilon \circ \theta_t f - f)^2] \, dm \\
&\leq\; \int (P_\varepsilon f^2 - 2 f P_\varepsilon f + f^2) \, dm \\
&\leq\; 2\|f\|_{L^2(m)} \|f - P_\varepsilon f\|_{L^2(m)}. \qquad (3.4)
\end{aligned}$$

Therefore, $\lim_{\varepsilon\to 0} \|S_{t+\varepsilon} f - S_t f\|_{L^2(m\otimes P)} = 0$, uniformly in t. □

Remark 3.3 *Note also the convergence in $L^2(m \otimes P)$ of $P_\varepsilon S_t f$ toward $S_t f$ when $\varepsilon \to 0$. Indeed $\|P_\varepsilon S_t f - S_t f\|^2_{L^2(m\otimes P)} = E[\|P_\varepsilon S_t f - S_t f\|^2_{L^2(m)}]$ and $\|P_\varepsilon S_t f - S_t f\|^2_{L^2(m)}$ converges towards 0 when ε goes to 0 and is dominated by $4\|S_t f\|^2_{L^2(m)}$.*

Proof of e). Let us remark that for any ε and t positive,

$$\begin{aligned}
S_{t+\varepsilon} f - S_t f \;&=\; S_t\left(P_\varepsilon f + \sum_k \int_0^\varepsilon S_u \circ \theta_t (D_{e_k} P_{\varepsilon-u} f) \, dW_u^k \circ \theta_t - f\right) \\
&=\; S_t(P_\varepsilon f - f) + \sum_k \int_t^{t+\varepsilon} S_s(D_{e_k} P_{t+\varepsilon-s} f) \, dW_s^k. \qquad (3.5)
\end{aligned}$$

Hence using (3.5) for $t = \frac{i}{n}t$ and $\varepsilon = \frac{1}{n}t$, for $f \in \mathcal{D}(A)$,

$$
\begin{aligned}
S_t f - f - \sum_k \int_0^t S_s(D_{e_k} f)\, dW_s^k - \int_0^t S_s(Af)\, ds \ &= \\
&= \sum_{i=0}^{n-1} \Bigg[S_{\frac{i}{n}t}(P_{\frac{t}{n}} f - f) + \sum_k \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} S_s(D_{e_k} P_{\frac{i+1}{n}t-s} f)\, dW_s^k \\
&\qquad - \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} S_s(Af)\, ds - \sum_k \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} S_s(D_{e_k} f)\, dW_s^k \Bigg] \\
&= A_1(n) + A_2(n) + A_3(n), \quad \text{with}
\end{aligned}
$$

$$
A_1(n) = \sum_{i=0}^{n-1} S_{\frac{i}{n}t}(P_{\frac{t}{n}} f - f - \frac{t}{n} Af); \tag{3.6}
$$

$$
A_2(n) = \sum_{i=0}^{n-1} \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} (S_{\frac{i}{n}t}(Af) - S_s(Af))\, ds; \tag{3.7}
$$

$$
A_3(n) = \sum_{i=0}^{n-1} \sum_k \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} S_s(D_{e_k}(P_{\frac{i+1}{n}t-s} f - f))\, dW_s^k. \tag{3.8}
$$

First, using the fact that m is P_t-invariant,

$$
\|A_1(n)\|_{L^2(m\otimes P)} \le n \|P_{\frac{t}{n}} f - f - \frac{t}{n} Af\|_{L^2(m)} = o(1). \tag{3.9}
$$

After, we remark that

$$
\left\| \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} (S_{\frac{i}{n}t}(Af) - S_s(Af))\, ds \right\|_{L^2(m\otimes P)}^2 \le \frac{t}{n} \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} \|S_{\frac{i}{n}t}(Af) - S_s(Af)\|_{L^2(m\otimes P)}^2\, ds.
$$

As $S_t(Af)$ is uniformly continuous in $L^2(m \otimes P)$, there exists $\varepsilon(x)$ such that $\lim_{x\to 0} \varepsilon(x) = 0$ and $\|S_{\frac{i}{n}t}(Af) - S_s(Af)\|_{L^2(m\otimes P)}^2 \le \varepsilon(\frac{t}{n})$ for any $s \in [\frac{i}{n}t, \frac{i+1}{n}t]$. Hence we get

$$
\left\| \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} (S_{\frac{i}{n}t}(Af) - S_s(Af))\, ds \right\|_{L^2(m\otimes P)}^2 \le \frac{t^2}{n^2} \varepsilon(\frac{t}{n})
$$

and $\|A_2(n)\|_{L^2(m\otimes P)} = o(1)$.

At last, as the different terms in the sum in equation (3.8) are orthogonal,

$$
\|A_3(n)\|_{L^2(m\otimes P)}^2 = \sum_{i=0}^{n-1} \sum_k \int E\left[\left(\int_{\frac{i}{n}t}^{\frac{i+1}{n}t} S_s(D_{e_k}(P_{\frac{i+1}{n}t-s} f - f)) dW_s^k \right)^2 \right] dm
$$

$$
\begin{aligned}
&\le \sum_{i=0}^{n-1} \int \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} |D(P_{\frac{i+1}{n}t-s} f - f)|^2 \, ds \, dm \\
&\le n \int_0^{\frac{t}{n}} \int |D(P_s f - f)|^2 \, dm \, ds \\
&\le n \int_0^{\frac{t}{n}} \mathcal{E}(P_s f - f, P_s f - f) \, ds.
\end{aligned}
$$

As $\lim_{s\to 0} \mathcal{E}(P_s f - f, P_s f - f) = 0$, $\|A_3(n)\|_{L^2(m\otimes P)} = o(1)$.

Taking the limit as n goes to ∞, this shows that $\|S_t f - f - \sum_k \int_0^t S_s(D_{e_k} f) \, dW_s^k - \int_0^t S_s(Af) \, ds\|_{L^2(m\otimes P)} = 0$. □

Proof that a), c) and e) imply d). Take $f \in L^2(m)$ and ε positive, assuming e),

$$
\begin{aligned}
&S_t P_\varepsilon f - P_t P_\varepsilon f - \sum_k \int_0^t S_s(D_{e_k} P_{t-s} P_\varepsilon f) \, dW_s^k = \\
&\quad = \sum_{i=0}^{n-1} \left[S_{\frac{i+1}{n}t}(P_{t-\frac{i+1}{n}t} P_\varepsilon f) - S_{\frac{i}{n}t}(P_{t-\frac{i}{n}t} P_\varepsilon f) - \sum_k \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} S_s(D_{e_k}(P_{t-s} P_\varepsilon f)) \, dW_s^k \right] \\
&\quad = B_1(n) + B_2(n) + B_3(n), \quad \text{with}
\end{aligned}
$$

$$
\begin{aligned}
B_1(n) &= \sum_{i=0}^{n-1} \sum_k \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} S_s(D_{e_k}(P_{t-\frac{i+1}{n}t} P_\varepsilon f - P_{t-s} P_\varepsilon f)) dW_s^k; \qquad (3.10) \\
B_2(n) &= -\sum_{i=0}^{n-1} S_{\frac{i}{n}t} \left(P_{t-\frac{i}{n}t} P_\varepsilon f - P_{t-\frac{i+1}{n}t} P_\varepsilon f - \frac{t}{n} A P_{t-\frac{i+1}{n}t} P_\varepsilon f \right); \qquad (3.11) \\
B_3(n) &= \sum_{i=0}^{n-1} \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} (S_s - S_{\frac{i}{n}t})(A P_{t-\frac{i+1}{n}t} P_\varepsilon f) \, ds, \qquad \text{since} \qquad (3.12) \\
S_{\frac{i+1}{n}t}(P_{t-\frac{i+1}{n}t} P_\varepsilon f) &= S_{\frac{i}{n}t}(P_{t-\frac{i+1}{n}t} P_\varepsilon f) + \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} S_s(A P_{t-\frac{i+1}{n}t} P_\varepsilon f) \, ds \\
&\quad + \sum_k \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} S_s D_{e_k} P_{t-\frac{i+1}{n}t} P_\varepsilon f \, dW_s^k.
\end{aligned}
$$

Since the different terms in the sum in equation (3.10) are orthogonal,

$$
\begin{aligned}
\|B_1(n)\|^2_{L^2(m\otimes P)} &\le \sum_{i=0}^{n-1} \int \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} |D(P_{t-\frac{i+1}{n}t} P_\varepsilon f - P_{t-s} P_\varepsilon f)|^2 \, ds \, dm \\
&\le \sum_{i=0}^{n-1} \int_{\frac{i}{n}t}^{\frac{i+1}{n}t} \mathcal{E}(P_{t-\frac{i+1}{n}t} P_\varepsilon f - P_{t-s} P_\varepsilon f, P_{t-\frac{i+1}{n}t} P_\varepsilon f - P_{t-s} P_\varepsilon f) \, ds \\
&\le n \int_0^{\frac{t}{n}} \mathcal{E}(P_s P_\varepsilon f - P_\varepsilon f, P_s P_\varepsilon f - P_\varepsilon f) \, ds \qquad (3.13)
\end{aligned}
$$

as $\mathcal{E}(P_t f, P_t f) \leq \mathcal{E}(f,f)$ for any positive t and any $f \in L^2(m)$.

Equation (3.13) implies that $\|B_1(n)\|_{L^2(m\otimes P)} = o(1)$ (as $\lim_{s\to 0}\mathcal{E}(P_sP_\varepsilon f - P_\varepsilon f, P_sP_\varepsilon f - P_\varepsilon f) = 0$).

$$\begin{aligned}
\|B_2(n)\|_{L^2(m\otimes P)} &\leq \sum_{i=0}^{n-1} \|S_{\frac{i}{n}t}\left(P_{t-\frac{i}{n}t}P_\varepsilon f - P_{t-\frac{i+1}{n}t}P_\varepsilon f - \frac{t}{n}AP_{t-\frac{i+1}{n}t}P_\varepsilon f\right)\|_{L^2(m\otimes P)} \\
&\leq \sum_{i=0}^{n-1}\left(\int\left(P_{\frac{i+1}{n}t}P_\varepsilon f - P_{\frac{i}{n}t}P_\varepsilon f - \frac{t}{n}AP_{\frac{i}{n}t}P_\varepsilon f\right)^2 dm\right)^{\frac{1}{2}} \\
&\leq n\|P_{\frac{t}{n}}P_\varepsilon f - P_\varepsilon f - \frac{t}{n}AP_\varepsilon f\|_{L^2(m)}.
\end{aligned}$$

Hence, $\|B_2(n)\|_{L^2(m\otimes P)} = o(1)$.

Note that if $Q_t f = E[S_t f]$, e) implies that for any $f \in \mathcal{D}(A)$,

$$Q_t f = f + \int_0^t Q_s(Af)ds.$$

Then $\frac{\partial}{\partial_s}Q_sP_{t-s}f = 0$ for any $f \in L^2(m)$ and $0 < s < t$ (then $P_{t-s}f \in \mathcal{D}(A)$) and we have $Q_t f = P_t f$. With this remark and the fact that a) and c) are satisfied, we see that b) and equation (3.4) are satisfied (see the proof of b). Using (3.4), we have

$$\begin{aligned}
&\|(S_s - S_{\frac{i}{n}t})(AP_{t-\frac{i+1}{n}t}P_\varepsilon f)\|^2_{L^2(m\otimes P)} \leq \\
&\qquad\leq 2\|AP_{t-\frac{i+1}{n}t}P_\varepsilon f\|_{L^2(m)}\|AP_{t-\frac{i+1}{n}t}P_\varepsilon f - P_{s-\frac{i}{n}t}AP_{t-\frac{i+1}{n}t}P_\varepsilon f\|_{L^2(m)} \\
&\qquad\leq 2\|AP_\varepsilon f\|_{L^2(m)}\|AP_\varepsilon f - P_{s-\frac{i}{n}t}AP_\varepsilon f\|_{L^2(m)} \\
&\qquad\leq 4\|AP_\varepsilon f\|^2_{L^2(m)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|B_3(n)\|^2_{L^2(m\otimes P)} &\leq \sum_{i=0}^{n-1}\frac{t}{n}\int_{\frac{i}{n}t}^{\frac{i+1}{n}t}\|(S_s - S_{\frac{i}{n}t})(AP_{t-\frac{i+1}{n}t}P_\varepsilon f)\|^2_{L^2(m\otimes P)}\,ds \\
&\leq \frac{4t^2}{n}\|AP_\varepsilon f\|^2_{L^2(m)}.
\end{aligned}$$

Taking the limit as n goes to ∞, this shows that d) is satisfied for $P_\varepsilon f$, with $f \in L^2(m)$ and ε positive.

At last, since $\|S_tP_\varepsilon f - S_t f\|_{L^2(m\otimes P)} \leq \|P_\varepsilon f - f\|_{L^2(m)}$ (because c) is satisfied), $\|P_{t+\varepsilon}f - P_t f\|_{L^2(m\otimes P)} \leq \|P_\varepsilon f - f\|_{L^2(m)}$ and $\|\sum_k\int_0^t S_s(D_{e_k}P_{t-s}(P_\varepsilon f - f))dW^k_s\|^2_{L^2(m\otimes P)} \leq t\mathcal{E}(P_\varepsilon f -$

$f, P_\varepsilon f - f)$. Taking the limit when ε goes to 0, we prove that d) is satisfied for any $f \in L^2(m)$. $\square$

Proof that S_t is Markovian.

A more concise proof of this fact has been given in [21], relying on Wiener exponentials and Girsanov formula. The advantage of the following proof is to be more explanatory, to give a relation with weak solutions and to yield a construction of the process law associated with the statistical solution S_t.

Let $(\Omega', \mathcal{G}, \mathcal{G}_t, X_t, P_x)$ be a Hunt process associated to $(\mathcal{E}, \mathcal{F})$ (see [11]), we will take a canonical version with $\Omega' = C(\mathbb{R}^+, X)$. Let $W_t = \sum_k W_t^k e_k$ be a cylindrical Brownian motion on H, independent of the Markov process X_t.

Let $\mathcal{M}$ be the space of the martingales additive functionals, $\mathcal{G}_t$–adapted such that if $M \in \mathcal{M}$, $E_x[M_t^2] < \infty$, $E_x[M_t] = 0$ q.e. and $e(M) < \infty$ where $e(M) = \sup_{t>0} \frac{1}{2t} E_m[M_t^2]$ (with $P_m = \int P_x \, dm(x)$). $(\mathcal{M}, e)$ is a Hilbert space (see [11]).

For $f \in \mathcal{F}$, $M^f \in \mathcal{M}$ denotes the martingale part of the semi-martingale $f(X_t) - f(X_0)$. For $g \in C_K(X) \subset L^2(\Gamma(f,f)dm)$[1], we note $g.M^f \in \mathcal{M}$ the martingale $\int_0^t g(X_s) \, dM_s^f$, then $\mathcal{M}_0 = \{\sum_{i=1}^n g_i.M^{f_i};\ n \in \mathbb{N},\ g_i \in C_K(X),\ f_i \in \mathcal{F}\}$ is dense in $\mathcal{M}$ (see lemma 5.6.3 in [11]), and $e(\sum_i g_i.M^{f_i}) = \frac{1}{2} \sum_{i,j} \int g_i g_j \Gamma(f_i, f_j) \, dm$ (see theorem 5.2.3 and 5.6.1 in [11]).

Lemma 3.4 *For every $(M, N) \in \mathcal{M} \times \mathcal{M}$, there exists $\Gamma(M, N) \in L^1(m)$ such that*

$$\langle M, N \rangle_t = \int_0^t \Gamma(M, N)(X_s) \, ds, \tag{3.14}$$

where $\langle ., . \rangle_t$ is the usual martingale bracket. And for $(f, g) \in \mathcal{F}$, $\Gamma(M^f, M^g) = \Gamma(f, g)$.

Note that lemma 3.4 implies that $e(M, N) = \frac{1}{2} \int \Gamma(M, N) \, dm$.

In the fundamental example 2.1, X_t is the Brownian motion on X, M_t^f is the Itô integral $\int_0^t \langle df(X_s), dX_s \rangle$, Γ is the inverse Riemannian metric and $\mathcal{M}$ can be identified with the space of 1-forms equipped with the L^2-norm associated with the metric.

Proof. When $f \in \mathcal{F}$, it follows from theorem 5.2.3 in [11] that

$$\langle M^f, M^f \rangle_t = \int_0^t \Gamma(f, f)(X_s) \, ds.$$

[1] $C_K(X)$ design the space of functions continuous with compact support.

For $M = \sum_i h_i.M^{f_i}$, $N = \sum_j k_j.M^{g_j}$, two martingales of $\mathcal{M}_0$,

$$\langle M, N\rangle_t = \sum_{i,j} \int_0^t h_i k_j \Gamma(f_i, g_j)(X_s)\, ds = \int_0^t \Gamma(M, N)(X_s)\, ds, \tag{3.15}$$

with $\Gamma(M, N) = \sum_{i,j} h_i k_j \Gamma(f_i, g_j)$. Γ is a bilinear mapping from $\mathcal{M}_0 \times \mathcal{M}_0$ in $L^1(m)$. Γ is continuous since for any $(M, N) \in \mathcal{M}_0 \times \mathcal{M}_0$,

$$\begin{aligned} \int |\Gamma(M, N)|\, dm &\leq \int \Gamma(M, M)^{\frac{1}{2}} \Gamma(N, N)^{\frac{1}{2}}\, dm \\ &\leq 2e(M)^{\frac{1}{2}} e(N)^{\frac{1}{2}}. \end{aligned}$$

It follows that Γ can be extended to $\mathcal{M} \times \mathcal{M}$.

Take $M \in \mathcal{M}$ and an approximating sequence $M_n \in \mathcal{M}_0$. Then $e(M_n - M)$ converges towards 0, M_n converges towards M in $L^2(P_x)$ and $\langle M_n, M_n\rangle_t$ converges in $L^1(P_x)$ towards $\langle M, M\rangle_t$ for almost every x (see section 5-2 in [11]). This proves that $\langle M, M\rangle_t = \int_0^t \Gamma(M, M)(X_s)\, ds$. □

Lemma 3.5 *If m is bounded, for any $h \in H$, there exists a unique continuous martingale in $\mathcal{M}$, N^h such that for any $f \in \mathcal{F}$, $e(N^h, M^f) = \frac{1}{2} \int D_h f\, dm$ and $\frac{d}{dt}\langle N^h, M^f\rangle_t = D_h f(X_t)$. In addition, $e(N^h) \leq \frac{1}{2} m(X) \|h\|^2$ and $\langle N^h\rangle_t \leq \|h\|^2 t$.*

In the Riemannian manifold case (example 2.1), $N_t^h = \int_0^t \langle h(X_s), dX_s\rangle$ when $\Gamma = D$.

Proof. For $h = \sum_k \lambda_k e_k \in H$, let us define a linear form, α_h on $\mathcal{M}_0$ such that for any $M = \sum_{i=1}^n g_i.M^{f_i} \in \mathcal{M}_0$, $\alpha_h(M) = \frac{1}{2} \sum_{i=1}^n \int g_i D_h f_i\, dm$.

$$\begin{aligned} (\alpha_h(M))^2 &= \left(\sum_k \lambda_k \frac{1}{2} \sum_{i=1}^n \int g_i D_{e_k} f_i\, dm \right)^2 \\ &\leq \frac{1}{4} \|h\|^2 m(X) \sum_{i,j} \int g_i g_j D(f_i, f_j)\, dm \leq \frac{1}{2} \|h\|^2 m(X) e(M). \end{aligned}$$

This proves that α_h is continuous on $\mathcal{M}_0$ and can be extended to a continuous linear form on $\mathcal{M}$ such that $\alpha_h(M) \leq \frac{1}{\sqrt{2}} \|h\| \sqrt{m(X)e(M)}$. To this form is associated a unique $N^h \in \mathcal{M}$ such that $\alpha_h(M) = e(N^h, M)$.

Note that for any $g \in C_K(X)$ and $f \in \mathcal{F}$, we have $\int g D_h f\, dm = \int \Gamma(N^h, g.M^f)\, dm = \int g \Gamma(N^h, M^f)\, dm$. This is satisfied for every $g \in C_K(X)$, therefore for any $f \in \mathcal{F}$, $\Gamma(N^h, M^f) = D_h f$.

Note that we also have, for $M \in \mathcal{M}_0$

$$\Gamma(N^h, M) \leq \|h\| \; \Gamma(M,M)^{\frac{1}{2}},$$

which implies that $\langle N^h \rangle_t \leq \|h\|^2 t$. $\square$

Remark 3.6 *When m is not bounded, N^h can be defined as a local martingale such that for any compact K and any $f \in \mathcal{F}$, $1_K.N^h \in \mathcal{M}$, $e(1_K.N^h, M^f) = \frac{1}{2}\int_K D_h f \; dm$. In addition, $e(1_K.N^h) \leq \frac{1}{2} m(K)\|h\|^2$.*

Let γ_{kl} be a function on X such that $\frac{d}{dt}\langle N^{e_l}, N^{e_k}\rangle_t = \gamma_{kl}(X_t)$. Lemma 3.5 implies that the matrix $A = ((\delta_{kl} - \gamma_{kl}))$ is positive (as $\frac{d}{dt}\langle N^h \rangle_t \leq \|h\|^2$). Therefore, it is possible to find a matrix R such that $R^2 = A$.

Remark 3.7 *If for any $f \in \mathcal{F}$, $\Gamma(f,f) = \|Df\|_H^2$, then for any $f \in \mathcal{F}_b$,*

$$M_t^f = \sum_k \int_0^t D_{e_k} f(X_s) dN_s^{e_k}, \tag{3.16}$$

$D_{e_k} f = \sum_l D_{e_l} f \; \gamma_{kl}(X_t)$ and the positive symmetric matrix $P = ((\gamma_{kl}))$ is a projector. In this case, $R = I - P$.

Proof. Set $Q_t^f = \sum_k \int_0^t D_{e_k} f(X_s) \; dN_s^{e_k}$, $Q^f \in \mathcal{M}$, then for any $M = \sum_{i=1}^n g_i . M^{f_i} \in \mathcal{M}_0$,

$$\begin{aligned}
\langle Q^f, M \rangle_t &= \sum_k \int_0^t D_{e_k} f(X_s) d\langle N^{e_k}, M\rangle_s \\
&= \sum_k \sum_{i=1}^n \int_0^t D_{e_k} f(X_s) g_i(X_s) D_{e_k} f_i(X_s) \; ds \\
&= \sum_{i=1}^n \int_0^t g_i(X_s) D(f, f_i)(X_s) \; ds \; = \; \langle M^f, M \rangle_t.
\end{aligned}$$

This implies that for any $M \in \mathcal{M}$, $e(Q^f, M) = e(M^f, M)$ and $Q^f = M^f$.

Since by lemma 3.5, $\frac{d}{dt}\langle M^f, N^{e_k}\rangle_t = D_{e_k} f(X_t)$, we get that

$$D_{e_k} f(X_t) = \frac{d}{dt}\langle Q^f, N^{e_k}\rangle_t = \sum_l D_{e_l} f(X_t) \frac{d}{dt}\langle N^{e_l}, N^{e_k}\rangle_t = \left(\sum_l D_{e_l} f \; \gamma_{kl}\right)(X_t).$$

This relation implies that $N_t^{e_k} = \sum_l \int_0^t \gamma_{kl}(X_s)\ dN_s^{e_l}$ (this is easy to check, considering $\frac{d}{dt}\langle N^{e_k}, M\rangle_t$ with $M \in \mathcal{M}_0$). From this, we see that $\gamma_{kl} = \sum_i \gamma_{ki}\gamma_{il}$ (i.e $P^2 = P$). □

Set $\widetilde{W}_t^k = N_t^{e_k} + \sum_l \int_0^t R_{kl}(X_s) dW_s^l$ and $\widetilde{W}_t = \sum_k \widetilde{W}_t^k e_k$.

In the Riemannian manifold case, when $\Gamma(f,f) = \|Df\|_H^2$ for any $f \in \mathcal{F}$, denoting C_ξ by $C_{(x,u)}$ when $u \in T_x X$ and $\xi = (x,u)$ we have :

$$\begin{aligned} d\widetilde{W}_t &= dW_t + C_{(X_t, dX_t)} - C_{(X_t, dW_t(X_t))} \\ \text{and} \quad d\widetilde{W}_t^k &= dW_t^k + \langle e_k(X_t), dX_t\rangle - \sum_l \langle e_k(X_t), e_l(X_t)\rangle\ dW_t^l. \end{aligned}$$

In this case, R is a projector (see remark above).

Lemma 3.8 $(\widetilde{W}_t^k)_k$ *is a sequence of independent Brownian motion.*

Proof. Since $\widetilde{W}_t^k$ is a continuous martingale, we just have to compute $\frac{d}{dt}\langle \widetilde{W}_t^k, \widetilde{W}_t^l\rangle_t$:

$$\frac{d}{dt}\langle \widetilde{W}_t^k, \widetilde{W}_t^l\rangle_t = \gamma_{kl} + R_{kl}^2 = \delta_{kl}.$$

This implies the lemma. □

Let μ be an initial distribution of the form hm, with h a positive function in $L^2(m) \cap L^1(m)$ and for $f \in L^2(m)$ define $\widetilde{S}_t f$ by the conditional expectation

$$\widetilde{S}_t f(X_0) = E_\mu[f(X_t) | \sigma(X_0, \widetilde{W}_s^k;\ k \in \mathbb{N};\ s \leq t)]. \tag{3.17}$$

(One checks easily that this definition does not depend on h.) Remark that as X_t is Markovian and W_t has independent increments,

$$\widetilde{S}_t f(X_0) = E_\mu[f(X_t) | \sigma(X_0, \widetilde{W}_s^k;\ k \in \mathbb{N};\ s \geq 0)]. \tag{3.18}$$

In the same way, we see that $\widetilde{S}_t$ satisfies the multiplicative cocycle property a).

Lemma 3.9 *For any $f \in \mathcal{D}(A)$ and μ an initial distribution absolutely continuous with respect to m,*

$$\widetilde{S}_t f = f + \sum_k \int_0^t \widetilde{S}_s(D_{e_k} f)\ d\widetilde{W}_s^k + \int_0^t \widetilde{S}_s(Af)\ ds, \quad P_\mu\ a.s.$$

Proof. For any $f \in \mathcal{D}(A)$, we have

$$f(X_t) = f(X_0) + M_t^f + \int_0^t Af(X_s)\ ds. \tag{3.19}$$

It is clear that $E_.[\int_0^t Af(X_s)\ ds|\sigma(\widetilde{W}_t^k;\ k \in \mathbb{N};\ s \leq t)] = \int_0^t \widetilde{S}_s Af(X_s)\ ds$, as (3.17) is satisfied. Let $Z_t = \sum_k \int_0^t H_s^k\ d\widetilde{W}_s^k \in L^2(\sigma(\widetilde{W}_s^k;\ k \in \mathbb{N};\ s \leq t))$,

$$\begin{aligned} E_.[Z_t M_t^f] &= \sum_k E_.\left[\int_0^t H_s^k\ d\langle \widetilde{W}^k, M^f\rangle_s\right] \\ &= \sum_k E_.\left[\int_0^t H_s^k D_{e_k} f(X_s)\ ds\right] \\ &= \sum_k E_.\left[\int_0^t H_s^k \widetilde{S}_s(D_{e_k} f)\ ds\right] \\ &= E_.\left[Z_t \sum_k \int_0^t \widetilde{S}_s(D_{e_k} f)\ d\widetilde{W}_s^k\right]. \end{aligned}$$

This proves that $E_.[M_t^f|\sigma(\widetilde{W}_t^k;\ k \in \mathbb{N};\ s \leq t)] = \sum_k \int_0^t \widetilde{S}_s D_{e_k} f\ d\widetilde{W}_s^k$. □

Now, using uniqueness in theorem 3.2 and the isomorphism j between $L^2(\sigma(\widetilde{W}_t^k;\ t \geq 0;\ k \in \mathbb{N}))$ and $L^2(\sigma(W_t^k;\ t \geq 0;\ k \in \mathbb{N}))$, we see that $j\widetilde{S}_t = S_t$, which implies that S_t is Markovian. □

Proposition 3.10 *For any $f \in \mathcal{F}_b$, the martingale*

$$P_t^f = M_t^f - \sum_k \int_0^t D_{e_k} f(X_s) d\widetilde{W}_s^k \tag{3.20}$$

is orthogonal to the family of martingales $\{\widetilde{W}_t^k;\ k \in \mathbb{N}\}$, in the sense of the martingale bracket (i.e for any k, $\langle P^f, \widetilde{W}^k\rangle_. = 0$). And for any $(f,g) \in \mathcal{F}_b^2$,

$$\langle P^f, P^g\rangle_t = \int_0^t (\Gamma(f,g)(X_s) - D(f,g)(X_s))\ ds. \tag{3.21}$$

Proof. We just have to show that $\langle P^f, \widetilde{W}^k\rangle_t = 0$ for every $f \in \mathcal{F}_b$ and every $k \in \mathbb{N}$ which is true as

$$\langle M^f, \widetilde{W}^k\rangle_t = \langle M^f, N^{e_k}\rangle_t = \int_0^t D_{e_k} f(X_s)\ ds.$$

Let $(f,g) \in \mathcal{F}_b^2$, then

$$\begin{aligned}
\langle P^f, P^g\rangle_t &= \langle P^f, M^g\rangle_t \\
&= \langle M^f, M^g\rangle_t - \sum_k \int_0^t D_{e_k} f(X_s) d\langle \widetilde{W}^k, M^g\rangle_s \\
&= \int_0^t \Gamma(f,g)(X_s)\ ds - \sum_k \int_0^t D_{e_k} f(X_s) D_{e_k} g(X_s)\ ds \\
&= \int_0^t \Gamma(f,g)(X_s)\ ds - \int_0^t D(f,g)(X_s)\ ds. \qquad \square
\end{aligned}$$

Remark 3.11 *In the case $\Gamma(f,f) = \|Df\|_H^2$ for any $f \in \mathcal{F}$, proposition 3.10 implies that $P_t^f = 0$ and that*

$$M_t^f = \sum_k \int_0^t D_{e_k} f(X_s) d\widetilde{W}_s^k.$$

From this, we see that the diffusion X_t satisfies the S.D.E.

$$f(X_t) - f(X_0) = \sum_l \int_0^t D_{e_l} f(X_s)\ d\widetilde{W}_s^k + \int_0^t Af(X_s)\ ds \tag{3.22}$$

for every $f \in \mathcal{D}(A)$. Therefore $(X_t, \widetilde{W}_t)$ appears as a weak solution of this S.D.E. and $\widetilde{S}_t$ is defined by filtering X_t with respect to $\widetilde{W}_t$.

Let $P_{x,\widetilde{\omega}}(d\omega')$ be the conditional law of the diffusion X_t, given X_0 and $\{\widetilde{W}_t;\ t \in \mathbb{R}^+\}$ (it is independent of the choice of the initial distribution). Using the identity in law between W and $\widetilde{W}$, we get a family of conditional probabilities $P_{x,\omega}(d\omega')$ on $C(\mathbb{R}^+, X)$ defined $m \otimes P$ a.e.

Remark that (with $X_t(\omega') = \omega'(t)$)

$$S_t f(x,\omega) = \int f(X_t(\omega'))\ P_{x,\omega}(d\omega') \qquad m \otimes P \quad a.s. \tag{3.23}$$

Under $P_{x,\omega}(d\omega')P(d\omega)$, $X_t(\omega')$ verifies the S.D.E. (3.3). It is a canonical weak solution of the S.D.E. (3.3) on a canonical extension of the probability space on which W is defined. S_t is obtained by filtering X_t with respect to W.

4 The n-point motion.

Let $P_t^{(n)}$ be the family of operators on $L^\infty(m^{\otimes n})$ such that, for any $(f_i)_{1\leq i\leq n} \in L^\infty(m)$,

$$P_t^{(n)} f_1 \otimes \ ... \ \otimes f_n = E[S_t f_1 \otimes \ ... \ \otimes S_t f_n]. \tag{4.1}$$

$P_t^{(n)}$ is a Markovian semigroup on $L^\infty(m^{\otimes n})$ as S_t is Markovian and satisfies a) in theorem 3.2. It is easy to check that $P_t^{(2)}$ maps tensor products of $L^2(m)$ functions into $L^2(m^{\otimes 2})$.

Proposition 4.1 *For any family of probability laws on X absolutely continuous with respect to m, $(\mu_i;\ 1 \leq i \leq n)$,*

$$P^{(n)}_{\mu_1,...,\mu_n}(d\omega'_1, ..., d\omega'_n) = \int_\Omega P(d\omega) \otimes_{i=1}^n P_{\mu_i,\omega}(d\omega'_i) \tag{4.2}$$

defines a Markov process on X^n (with initial distribution $\otimes_{i=1}^m \mu_i$) associated with $P_t^{(n)}$. We shall call this Markov process on X^n the n-point motion.

Proof. For every family of functions in $L^\infty(m)$, $(f_i)_{\ 1\leq i\leq n}$, $m^{\otimes n} \otimes P$ a.e (with $X_t^i(\omega'_i) = \omega'_i(t)$)

$$\begin{aligned} S_t^{\otimes n} f_1 \otimes ... \otimes f_n(x_1, ..., x_n, \omega) &= \prod_{i=1}^n S_t f_i(x_i, \omega) \\ &= \int \prod_{i=1}^n f_i(X_t^i(\omega'_i)) \otimes_{i=1}^n P_{x_i,\omega}(d\omega'_i). \end{aligned} \tag{4.3}$$

We get the result by integrating both members of (4.3) with respect to $P(d\omega)$. □

Let $D^{(n)}$ be the linear map from $H \times \mathcal{F}^{\otimes n}$ in $L^2(m^{\otimes n})$ such that for any $(f_i)_{1\leq i\leq n} \in \mathcal{F}$ and $h \in H$,

$$D_h^{(n)} f_1 \otimes ... \otimes f_n = \sum_{i=1}^n f_1 \otimes ... \otimes D_h f_i \otimes ... \otimes f_n. \tag{4.4}$$

Proposition 4.2 *For any $(f_i)_{1\leq i\leq n} \in \mathcal{D}(A) \cap L^\infty(m)$,*

$$\begin{aligned} S_t^{\otimes n} f_1 \otimes ... \otimes f_n &= f_1 \otimes ... \otimes f_n + \sum_k \int_0^t S_s^{\otimes n}(D_{e_k}^{(n)} f_1 \otimes ... \otimes f_n)\ dW_s^k \\ &\quad + \int_0^t S_s^{\otimes n}(A^{(n)} f_1 \otimes ... \otimes f_n)\ ds, \end{aligned}$$

where

$$
\begin{aligned}
A^{(n)} f_1 \otimes ... \otimes f_n &= \sum_{i=1}^{n} f_1 \otimes ... \otimes A f_i \otimes ... \otimes f_n \\
&+ \sum_{1 \le i < j \le n} \sum_k f_1 \otimes ... \otimes D_{e_k} f_i \otimes ... \otimes D_{e_k} f_j \otimes ... \otimes f_n.
\end{aligned}
$$

Remark. 1) For $n = 2$, the formula extends to functions in $\mathcal{D}(A)$ and $A^{(2)} f \otimes g = Af \otimes g + f \otimes Ag + C(f,g)$, where $(f,g) \in (\mathcal{D}(A))^2$.

2) Taking the expectation, we see that $A^{(n)}$ is the infintesimal generator of $P_t^{(n)}$ on $(\mathcal{D}(A) \cap L^\infty(m))^{\otimes 2}$.

3) The formula extends to $C_K^2(X^n)$ in the Riemannian manifold case (using for example the uniform density of sums of product functions and the regularizing effect of $P_\varepsilon^{\otimes n}$).

Proof. This is just a straightforward application of Itô's formula applied to $S_t f_1 \otimes ... \otimes S_t f_n$, using the differential form of the equation satisfied by S_t, $e)$ in theorem 3.2. Taking the expectation and differentiating with respect to t, we get

$$
\begin{aligned}
\frac{d}{dt}_{|t=0} P_t^{(n)} f_1 \otimes ... \otimes f_n &= \frac{d}{dt}_{|t=0} E[S_t^{\otimes n} f_1 \otimes ... \otimes f_n] \\
&= A^{(n)} f_1 \otimes ... \otimes f_n. \qquad \square
\end{aligned}
$$

Remark 4.3 *In general, $m^{\otimes n}$ is not invariant under $P_t^{(n)}$.*

5 Measure preserving case.

We say that the statistical solution S_t is measure preserving if and only if $mS_t = m$ a.s for all t (i.e m is invariant for S_t). When $m(X) = \infty$, we use the natural extension of S_t to $L^1(m)$ or to positive functions defined m–a.e.

Let us denote by $\mathcal{F}_K$ the set of functions of $\mathcal{F}$ which have compact support.

Proposition 5.1 *S_t is measure preserving if and only if $\int C(f,g)\, dm^{\otimes 2}$ vanishes for all f, g in $\mathcal{F}_K$. Moreover, define r_t on $L^2(\mathcal{F}_t)$ by $W_s^k \circ r_t = W_{t-s}^k - W_t^k$. Then the adjoint of S_t in $L^2(m)$ is $S_t^* = S_t \circ r_t$.*

Remark 5.2 *a) When $f \in \mathcal{F}_K$, $C(f,f) \in L^1(m^{\otimes 2})$.*

b) In the Riemannian manifold case, the condition that $\int C(f,g)\ dm^{\otimes 2}$ vanishes for all f, g in $\mathcal{F}_K$ is equivalent to assume that W_t is divergent free in the weak sense, i.e that for any $f \in \mathcal{F}_K$, $\int \langle W_t, \nabla f\rangle\ dm = 0$. (It follows from the identity $E\left[\left(\int \langle W_t, \nabla f\rangle\ dm\right)^2\right] = t\int C(f,f)\ dm^{\otimes 2}$.)

Lemma 5.3 *Assume that $\int C(f,g)\ dm^{\otimes 2}$ vanishes for all f, g in $\mathcal{F}_K$, then for every $h \in H$, f, g in $\mathcal{F}$,*

$$\int gD_h f\ dm = -\int fD_h g\ dm. \tag{5.1}$$

Proof. For every $h \in H$, $(g,f) \mapsto \int gD_h f\ dm$ is a continuous bilinear form on $\mathcal{F} \times \mathcal{F}$ since $\|D_h f\|^2_{L^2(m)} \le \mathcal{E}(f,f)\|h\|^2_{L_2(m)}$.

Take f, g in $\mathcal{F}_K \cap L^\infty(m)$ then $fg \in \mathcal{F}_K$ (as the bounded functions of a Dirichlet space form an algebra) and, since D_{e_k} is a derivation, $D_{e_k}(fg) = gD_{e_k}f + fD_{e_k}g$. Using this property, we get

$$\begin{aligned}\sum_k \left(\int (gD_{e_k}f + fD_{e_k}g)\ dm\right)^2 &= \sum_k \left(\int D_{e_k}(fg)\ dm\right)^2 \\ &= \int C(fg,fg)\ dm^{\otimes 2} = 0.\end{aligned}$$

This implies that for every k, $\int gD_{e_k}f\ dm = -\int fD_{e_k}g\ dm$. To conclude we observe that both members of (5.1) are continuous in f and g and that $\mathcal{F}_K \cap L^\infty(m)$ is dense in $\mathcal{F}$ (since the Dirichlet form is regular, see section 1.1 in [11]). $\square$

Proof of proposition 5.1. Assume $\int C(f,g)\ dm^{\otimes 2} = 0$ holds for every f and g in $\mathcal{F}_K$.

Let us remark that the expression of the n-th chaos of $S_t f$ is given by the expression

$$J^n_t f = \int_{0\le s_1\le s_2\le \ldots \le s_n \le t} \sum_{k_1,\ldots,k_n} P_{s_1} D_{e_{k_1}} P_{s_2-s_1} D_{e_{k_2}} \ldots D_{e_{k_n}} P_{t-s_n} f\ dW^{k_1}_{s_1}\ldots dW^{k_n}_{s_n}. \tag{5.2}$$

From this expression, using lemma 5.3 and the fact that P_t is self–adjoint in $L^2(m)$, we get that for f and g in $L^2(m)$,

$$\begin{aligned}\int gJ^n_t f\ dm \ &= \\ &= \int \int_{0\le s_1 \ldots \le s_n \le t} f \sum_{k_1,\ldots,k_n} (-1)^n P_{t-s_n} D_{e_{k_n}} P_{s_n - s_{n-1}} \ldots P_{s_2-s_1} D_{e_{k_1}} P_{s_1} g\ dW^{k_1}_{s_1}\ldots dW^{k_n}_{s_n}\ dm.\end{aligned} \tag{5.3}$$

Making the change of variable $u_{n-i+1} = t - s_i$, we get that the adjoint of J_t^n is given by

$$(J^n)_t^* g = \int_{0\le u_1\le u_2\le \ldots \le u_n \le t} \sum_{k_1,\ldots,k_n} P_{u_1} D_{e_{k_1}} P_{u_2-u_1} D_{e_{k_2}} \ldots D_{e_{k_n}} P_{t-u_n} g \, dW_{u_1}^{k_1} \circ r_t \ldots dW_{u_n}^{k_n} \circ r_t. \quad (5.4)$$

From this it is easy to see that $S_t^* g = (S_t \circ r_t) g$ (as they have the same chaos expansion).

Notice that $S_t^* 1 = 1$. A priori the constant functions are not in $L^2(m)$, but there exists an increasing sequence in $L^2(m)$, g_n such that g_n converges towards 1. For any nonnegative function $f \in L^2(m)$,

$$\int S_t f g_n \, dm = \int f S_t^* g_n \, dm. \quad (5.5)$$

This equation implies, taking the limit as n goes to ∞, that

$$m S_t(f) = \int f S_t^* 1 \, dm = m(f). \quad (5.6)$$

And we get that $mS_t = m$ a.s. Which ends the first part of the proof.

Conversely, it follows from proposition 4.2 that for all f, g in $\mathcal{D}(A)$,

$$S_t^{\otimes 2} f \otimes g - S_t f \otimes g - f \otimes S_t g + f \otimes g - \int_0^t S_s^{\otimes 2} C(f,g) \, ds$$

is a square integrable martingale. This result extends to f, g in $\mathcal{F}$. Taking f, g in $\mathcal{F}_K$, integrating with respect to $m^{\otimes 2}$ and taking expectation, we get that $\int C(f,g) \, dm^{\otimes 2}$ vanishes.

□

Remark 5.4 *When S_t is measure preserving, $P_t^{(n)}$ is self–adjoint in $L^2(m^{\otimes n})$ and in particular $m^{\otimes n}$ is invariant under $P_t^{(n)}$. The associated local Dirichlet form $\mathcal{E}^{(2)}$ is such that*

$$\mathcal{E}^{(2)}(f \otimes g, f \otimes g) = \mathcal{E}(f,f) \|g\|_{L^2(m)}^2 + \mathcal{E}(g,g) \|f\|_{L^2(m)}^2 + 2 \int C(f,g) f \otimes g \, dm^{\otimes 2}$$

for any $(f,g) \in \mathcal{F}^2$ and a similar expression can be given for $\mathcal{E}^{(n)}$.

6 Existence of a flow of maps.

Let $(S_t)_{t\ge 0}$ denote the statistical solution.

Definition 6.1 *We say that $(S_t)_{t\ge 0}$ is a flow of maps if and only if there exists a family of measurable mappings $(\varphi_t)_{t\ge 0}$ from $X \times \Omega$ in X such that for any $f \in L^2(m)$ and any positive t, $S_t f = f \circ \varphi_t$.*

Note that if $(S_t)_{t\geq 0}$ is a flow of maps, $P_{x,w}$ is the Dirac measure on the path $\{\varphi_t(x);\ t \geq 0\}$.

Definition 6.2 *We say that $(S_t)_{t\geq 0}$ is a coalescent flow of maps if and only if $(S_t)_{t\geq 0}$ is a flow of maps and for every $(x,y) \in X^2$, with positive probability there exists T such that $\varphi_t(x) = \varphi_t(y)$ for all $t \geq T$.*

let $((X_t, Y_t))_{t\geq 0}$ design the two–point motion associated to the statistical solution.

Definition 6.3 *We say that $(S_t)_{t\geq 0}$ is diffusive without hitting if and only if $(S_t)_{t\geq 0}$ is not a flow of maps and starting from (x,x), for all positive t, $X_t \neq Y_t$.*

Definition 6.4 *We say that $(S_t)_{t\geq 0}$ is diffusive with hitting if and only if $(S_t)_{t\geq 0}$ is not a flow of maps and $(X_t, Y_t)_{t\geq 0}$ hits the diagonal with positive probability.*

In this section, we will give conditions under which the statistical solution is a flow of maps or not.

Lemma 6.5 *$(S_t)_{t\geq 0}$ is a flow of maps if and only if for any $f \in L^2(m)$ and any positive t, $E[(S_t f)^2] = P_t f^2$.*

Proof. It is clear that there exists Markovian kernels on X, $s_t(x,\omega,dy)$ such that $S_t f(x) = \int f(y) s_t(x,\omega,dy)$. And $s_t(x,\omega,dy)$ is the law of $X_t(\omega')$ under $P_{x,\omega}(d\omega')$. As $m \otimes P$–a.e,

$$(S_t f^2)(x) - (S_t f)^2(x) = \int \left(f(y) - \int f(z) s_t(x,\omega,dz) \right)^2 s_t(x,\omega,dy), \tag{6.1}$$

if $E[(S_t f)^2] = P_t f^2$, $\int (f(y) - \int f(z) s_t(x,\omega,dz))^2 s_t(x,\omega,dy) = 0$ and $s_t(x,\omega,dz)$ is a Dirac measure $\delta_{\varphi_t(x,\omega)}$, where $\varphi_t(x,\omega)$ is defined $m \otimes P$–a.e. $\square$

Let $h \in L^1(m)$ be a positive function such that $\int h\ dm = 1$. For any positive t, let μ_t be a probability on the Borel sets of $X \times X$ such that for any $(f,g) \in L^2(m) \times L^2(m)$, $\mu_t(f \otimes g) = \int E[S_t f S_t g]\ h\ dm$.

Remark 6.6 *$(S_t)_{t\geq 0}$ is a flow of maps if and only if for all positive t, $\mu_t(\Delta) = 1$, where $\Delta = \{(x,x);\ x \in X\}$.*

Proof. If $(S_t)_{t\geq 0}$ is a flow of maps, there exists φ_t such that $S_t f = f \circ \varphi_t$. If A and B are disjoints Borel sets of finite measure,

$$\mu_t(A \times B) = \int E[1_A(\varphi_t(x))1_B(\varphi_t(x))]\ h(x)\ dm(x) = 0.$$

This implies that $\mu_t(X \times X - \Delta) = 0$ and as μ_t is a probability that $\mu_t(\Delta) = 1$.

If $\mu_t(\Delta) = 1$, for $f \in L^2(m)$, $\mu_t(f^2 \otimes 1 - 2f \otimes f + 1 \otimes f^2) = 0$. This implies that

$$\int_X P_t f^2\ h\ dm = \int_X E[(S_t f)^2]\ h\ dm \tag{6.2}$$

and that $E[(S_t f)^2] = P_t f^2$. Hence $(S_t)_{t\geq 0}$ is a flow of maps. □

Recall that we denoted by $P^{(2)}_{(.,.)}$ the law of the two–point motion $((X_t, Y_t))_{t\geq 0}$.

Proposition 6.7 *$(S_t)_{t\geq 0}$ is a flow of maps if for any positive r and any positive t,*

$$\lim_{y\to x} P^{(2)}_{(x,y)}[d(X_t, Y_t) \geq r] = 0 \quad m(dx) - a.e.$$

Proof. For $\varepsilon > 0$, let ν_ε be the measure on $X \times X$ such that for any $(f, g) \in L^2(m) \times L^2(m)$, $\nu_\varepsilon(f \otimes g) = \int f\ P_\varepsilon g\ h\ dm$. For any $(f, g) \in L^2(m) \times L^2(m)$,

$$\nu_\varepsilon P^{(2)}_t(f \otimes g) = \int E[S_t f P_\varepsilon S_t g]\ h\ dm. \tag{6.3}$$

As $P_\varepsilon S_t g$ converges in $L^2(m \otimes P)$ towards $S_t g$ (see remark 3.3),

$$\lim_{\varepsilon\to 0} \nu_\varepsilon P^{(2)}_t(f \otimes g) = \int E[S_t f S_t g]\ h\ dm. \tag{6.4}$$

Therefore, the family of measure $(\nu_\varepsilon P^{(2)}_t)_{\varepsilon>0}$ converges weakly as ε goes to 0 towards μ_t.

Assume that for any positive r and any t, $\lim_{y\to x} P^{(2)}_{(x,y)}[d(X_t, Y_t) \geq r] = 0$. Let A and B be two disjoint Borel sets such that $d(A, B) \geq r$, then

$$\nu_\varepsilon P^{(2)}_t(A \times B) = \int_X f_\varepsilon(x)\ h(x)\ dm(x),$$

with

$$f_\varepsilon(x) = \int P^{(2)}_{(x,y)}[X_t \in A \text{ and } Y_t \in B]\ p_\varepsilon(x, dy),$$

where $p_\varepsilon(x, dy)$ is the kernel given by P_ε.

As $d(A,B) \geq r$,

$$f_\varepsilon(x) \leq \int P^{(2)}_{(x,y)}[d(X_t,Y_t) \geq r]\; p_\varepsilon(x,dy).$$

For any positive β, for m almost every x, there exists $\alpha(x)$ such that $d(x,y) \leq \alpha(x)$ implies that $P^{(2)}_{(x,y)}[d(X_t,Y_t) \geq r] \leq \beta$. Note that

$$f_\varepsilon(x) \leq \int_{\{d(x,y)>\alpha(x)\}} p_\varepsilon(x,dy) + \int_{\{d(x,y)\leq\alpha(x)\}} P^{(2)}_{(x,y)}[d(X_t,Y_t) \geq r]\; p_\varepsilon(x,dy).$$

It is clear that $\lim_{\varepsilon\to 0} \int_{\{d(x,y)>\alpha(x)\}} p_\varepsilon(x,dy) = 0$ $m(dx)$–a.e. Hence, $\limsup f_\varepsilon(x) \leq \beta$ $m(dx)$–a.e and this holds for any positive β. Therefore, $\lim_{\varepsilon\to 0} f_\varepsilon(x) = 0$ $m(dx)$–a.e and by dominated convergence ($|f_\varepsilon(x)| \leq 1$) that

$$\lim_{\varepsilon\to 0} \nu_\varepsilon P^{(2)}_t(A\times B) = 0.$$

This implies that $\mu_t(X\times X - \Delta) = 0$ and that $(S_t)_{t\geq 0}$ is a flow of maps. □

Proposition 6.8 *If there exists a positive t, a positive r and $p \in]0,1]$ such that for $m^{\otimes 2}$ almost every (x,y), $P^{(2)}_{(x,y)}[d(X_t,Y_t) > r] \geq p$, then $(S_t)_{t\geq 0}$ is not a flow of maps.*

Proof. Suppose there exists a positive t, a positive r and $p \in]0,1]$ such that for $m^{\otimes 2}$ almost every (x,y), $P^{(2)}_{(x,y)}[d(X_t,Y_t) > r] \geq p$.

Let $(B_i)_{i\in\mathbb{N}}$ be a partition of X such that the diameter of B_i is lower than r.

Let us suppose that $\mu_t(\Delta) = 1$ (or that $(S_t)_t$ is a flow of maps). Then we have $\sum_i \mu_t(B_i \times B_i) = 1$ and for any positive α, there exists N such that

$$\sum_{i=1}^N \mu_t(B_i\times B_i) \geq 1-\alpha.$$

Since $\nu_\varepsilon P^{(2)}_t$ converges weakly towards μ_t,

$$\begin{aligned}
\sum_{i=1}^N \mu_t(B_i\times B_i) &= \lim_{\varepsilon\to 0} \nu_\varepsilon P^{(2)}_t(B_i\times B_i) \\
&= \lim_{\varepsilon\to 0} \sum_{i=1}^N \int_{X\times X} P^{(2)}_{(x,y)}[(X_t,Y_t)\in B_i\times B_i]\; p_\varepsilon(x,dy)\; h(x)dm(x) \\
&\leq \lim_{\varepsilon\to 0} \sum_{i=1}^N \int_{X\times X} P^{(2)}_{(x,y)}[X_t\in B_i;\; d(X_t,Y_t)\leq r]\; p_\varepsilon(x,dy)\; h(x)dm(x) \\
&\leq \lim_{\varepsilon\to 0} \int_{X\times X} P^{(2)}_{(x,y)}[d(X_t,Y_t)\leq r]\; p_\varepsilon(x,dy)\; h(x)dm(x) \leq 1-p
\end{aligned}$$

Choosing $\alpha < p$, we get a contradiction. Hence $\mu_t(\Delta) < 1$ and $(S_t)_{t\geq 0}$ is not a flow of maps. $\square$

7 A one dimensional example.

Let $X = \mathbb{R}$, P_t be the semigroup of the Brownian motion on $\mathbb{R}$ and the covariance function $C(x,y) = \mathrm{sgn}(x)\mathrm{sgn}(y)$ (where $\mathrm{sgn}(x)$ denotes the sign of x with the convention $\mathrm{sgn}(0) = 1$). Here, we have $W_t(x) = \mathrm{sgn}(x)W_t$, where W_t is a Brownian motion starting from 0. Set $L_t^x = \sup_{s\leq t}\{-\mathrm{sgn}(x)(x + W_s)\} \vee 0$ and $R_t^x = x + W_t + \mathrm{sgn}(x)L_t^x$ (it is a Brownian motion starting from x, reflected at 0).

Proposition 7.1 *The statistical solution S_t can be written the following way*

$$S_t f(x) = f(R_t^x)1_{L_t^x=0} + \frac{1}{2}\left[f(R_t^x) + f(-R_t^x)\right]1_{L_t^x>0}. \tag{7.1}$$

Proof. On an extension of the probability space, it is possible to build a Brownian motion starting from x, X_t such that $W_t = \int_0^t \mathrm{sgn}(X_s)dX_s$ (then X_t is a weak solution of the S.D.E. $dX_t = \mathrm{sgn}(X_t)dW_t$). Then $S_t f(x) = E[f(X_t)|\mathcal{F}^B]$, with $\mathcal{F}^B = \sigma(W_u; u \geq 0)$. Let us remark that L_t^x is the local time of X at 0 and that $R_t^x = \mathrm{sgn}(x)|X_t|$. Set $T = \inf\{t;\ L_t^x > 0\} = \inf\{t;\ X_t = 0\}$. The formula (7.1) follows simply from the fact that

$$E[f(X_t)1_{t\geq T} \mid |X_t|] = \frac{1}{2}\left(f(X_t) + f(-X_t)\right)1_{t\geq T}. \quad \square$$

8 The Lipschitz case.

Assume X is a Riemannian manifold with injectivity radius $\rho > 0$. Let P_t be the semigroup of a symmetric diffusion on X with generator A. Let C be a covariance inducing the metric (i.e with equality in (1.7)).

We will say that C is Lipschitz if and only if there exist a positive constant k and $0 < \varepsilon < \rho$ such that : For any $(x,y) \in X^2$, with $d(x,y) < \varepsilon$,

$$A^{(2)}d^2(x,y) \leq k\ d^2(x,y). \tag{8.1}$$

Remark. – $d^2(x,y)$ is smooth on $\{(x,y) \in X^2,\ d(x,y) < \rho\}$ since ρ is the injectivity radius.

– On $\mathbb{R}^d$, the condition (8.1) will be checked as soon as $A = \frac{1}{2}\sum_{1\le i,j\le d}^{d} C^{ij}(x,x)\partial_i\partial_j + \sum_i b^i(x)\partial_i$,

$$\sum_{i=1}^{d}(C^{ii}(x,x) + C^{ii}(y,y) - 2C^{ii}(x,y)) \le \frac{k}{2}\ d(x,y)^2 \tag{8.2}$$

and b^i is a Lipschitz function for all i.

Equation (8.2) is satisfied when C is C^2 or when $C = \sum_{\alpha=1}^{n} X_\alpha \otimes X_\alpha$, where X_α are Lipschitz vector fields. In the latest case, the flow of maps can be constructed by the usual fixed point method for solutions of S.D.E.'s based on Gronwall's lemma.

Let (X_t, Y_t) be the two–point motion associated with the statistical solution. Set $\tau = \inf\{t,\ d(X_t, Y_t) \ge \varepsilon\}$ and $H_t = d^2(X_{t\wedge\tau}, Y_{t\wedge\tau})$.

Lemma 8.1 *$E^{(2)}_{(x,y)}(H_t) \le e^{kt}\ d^2(x,y)$.*

Proof. By Itô's formula,

$$H_t - H_0 = M_t + \int_0^{t\wedge\tau} A^{(2)} d^2(X_s, Y_s)\ ds$$

where M_t is a martingale. Hence

$$\begin{aligned} H_t - H_0 &\le M_t + \int_0^{t\wedge\tau} k\ d^2(X_s, Y_s)\ ds \\ &\le M_t + \int_0^t k\ H_s\ ds. \end{aligned}$$

This implies that $E^{(2)}_{(x,y)}(H_t) - d^2(x,y) \le k\int_0^t E^{(2)}_{(x,y)}(H_s)\ ds$. Hence the lemma. □

Theorem 8.2 *Assume (8.1) is satisfied then the statistical solution associated to P_t and C is a flow of maps.*

Proof. Indeed, for any $r < \varepsilon$,

$$P^{(2)}_{(x,y)}[d(X_t, Y_t) \ge r] \le P^{(2)}_{(x,y)}[d(X_t, Y_t) \ge r \text{ or } t \ge \tau] \le \frac{1}{r^2}E^{(2)}_{(x,y)}(H_t) \le \frac{e^{kt}}{r^2}d(x,y)^2,$$

which goes to 0 as $d(x,y)$ goes to 0. And we conclude using theorem 6.7. □

9 Isotropic statistical solution on S^d.

9.1 Isotropic covariance function on S^d.

On S^d with $d \geq 2$, the isotropic covariance function C are given by the formula (see Raimond [25])

$$C((x,u),(y,v)) = \alpha(t)\langle u,v\rangle + \beta(t)\langle u,y\rangle\langle v,x\rangle, \tag{9.1}$$

with $(x,y) \in S^d \times S^d$, $t = \langle x,y\rangle = \cos\varphi$ and $(u,v) \in T_xS^d \times T_yS^d$. α and β are given by

$$\alpha(t) = \sum_{l=1}^{\infty} a_l\gamma_l(t) + \sum_{l=1}^{\infty} b_l \left(t\gamma_l(t) - \frac{1-t^2}{d-1}\gamma_l'(t)\right), \tag{9.2}$$

$$\beta(t) = \sum_{l=1}^{\infty} a_l\gamma_l'(t) + \sum_{l=1}^{\infty} b_l \left(-\gamma_l(t) - \frac{t}{d-1}\gamma_l'(t)\right), \tag{9.3}$$

where $\gamma_l(t) = C_{l-1}^{\frac{d+1}{2}}(t)/C_{l-1}^{\frac{d+1}{2}}(1)$, C_l^p is a Gegenbauer polynomial, a_l and b_l are nonnegative such that $\sum_l a_l < \infty$ and $\sum_l b_l < \infty$. Using the integral form of the Gegenbauer polynomials (see [27] p. 496):

$$\gamma_l(\cos\varphi) = \int_0^{\pi} [z(\varphi,\theta)]^{l-1} \sin^d\theta \, \frac{d\theta}{c_d}, \tag{9.4}$$

with $c_d = \int_0^\pi \sin^d\theta \, d\theta$ and $z(\varphi,\theta) = \cos\varphi - i\sin\varphi\cos\theta$.

In [12], it is proved that the spectrum of the Laplacian Δ acting on the L^2–vector fields is $\{-l(l+d-1),\ l \geq 1\} \cup \{-(l+1)(l+d-2),\ l \geq 1\}$. Let $\mathcal{G}_l$ and $\mathcal{D}_l$ be respectively the eigenspaces corresponding to the eigenvalues $-l(l+d-1)$ and $-(l+1)(l+d-2)$. $\mathcal{G}_l$ is constitued of gradient vector fields and $\mathcal{D}_l$ of divergent free vector fields. These spaces can be isometrically identified with the spaces $\mathcal{H}_{d+1,l}$ and $\mathcal{F}_{d+1,l}$ used in [25] and can be used as carrier spaces of the irreducible representations of $SO(d+1)$, T^l and Q^l.

Let $(\alpha_M^l)_M$ and $(\omega_M^l)_M$ be orthonormal basis of $\mathcal{G}_l$ and $\mathcal{D}_l$. Then, if $(z_{M,d}^l)_{l,M}$ and $(z_{M,\delta}^l)_{l,M}$ are independent families of independent normalized centered Gaussian variables,

$$W = \sum_{l\geq 1} \sqrt{\frac{d\ a_l}{\dim\mathcal{G}_l}} \sum_M z_{M,d}^l \alpha_M^l + \sum_{l\geq 1} \sqrt{\frac{d\ b_l}{\dim\mathcal{D}_l}} \sum_M z_{M,\delta}^l \omega_M^l \tag{9.5}$$

is an isotropic Gaussian vector fields of covariance C given by (9.1), (9.2) and (9.3).

Sketch of proof. The covariance of W is

$$\sum_{l\geq 1} \frac{d\ a_l}{\dim\mathcal{G}_l} \sum_M \alpha_M^l \otimes \alpha_M^l + \sum_{l\geq 1} \frac{d\ b_l}{\dim\mathcal{D}_l} \sum_M \omega_M^l \otimes \omega_M^l.$$

Let us choose $(\alpha^l_M)_M$ such that $\alpha^l_M = c_1(l,d)\nabla\Xi^l_M$ (where $(\Xi^l_M)_M$ is the basis of $\mathcal{H}_{d+1,l}$ given in [25]). Then, using the fact that $\Xi^l_M(p) = 0$ if $M \neq 0$, for $x = g_1 p$ and $y = g_2 p$ (with $p = (0, ..., 0, 1)$),

$$\begin{aligned}\sum_M \Xi^l_M(x)\Xi^l_M(y) &= \sum_{M,N,K} T^l_{MN}(g_1)T^l_{MK}(g_2)\Xi^l_N(p)\Xi^l_K(p) \\ &= T^l_{00}(g_2^{-1}g_1)(\Xi^l_0(p))^2.\end{aligned}$$

In [27] and [25], $T^l_{00}(g)$ is computed and it is easy from this to give the covariance of the gradient part of W. We can calculate the covariance of the divergent free part in a similar way : We choose the orthonormal basis $(\omega^l_M)_M$ of $\mathcal{D}_l$ such that for $M \notin \{1, ..., d\}$, $\omega^l_M(p) = 0$ and such that for $1 \leq i \leq d$, $\omega^l_i(p) = c_2(l,d)e_i$ (this basis corresponds to the basis of $\mathcal{F}_{d+1,l}$ given in [25]). Then one have for $x = gp$ and $g \in SO(d)$,

$$\omega^l_M(x) = \sum_{i=1}^{d} Q^l_{Mi}(g)g(\omega^l_i(p)) = c_2(l,d)Q^l_{Mi}(g)g(e^i). \tag{9.6}$$

Then for every (x, u) and (y, v) in TS^d,

$$\begin{aligned}\sum_M \langle\omega^l_M(x), u\rangle\langle\omega^l_M(y), v\rangle &= (c_2(l,d))^2 \sum_M Q^l_{Mi}(g_1)Q^l_{Mj}(g_2)\langle g_1(e^i), u\rangle\langle g_2(e^j), v\rangle \quad (9.7)\\ &= (c_2(l,d))^2 Q^l_{ji}(g)\langle g_1(e^i), u\rangle\langle g_2(e^j), v\rangle, \quad (9.8)\end{aligned}$$

with $g = g_2^{-1}g_1$. In [25], the matrix elements $Q^l_{ji}(g)$ are calculated and it is easy from this to give the covariance of the divergence free part of W. $\square$

Let us now introduce Sobolev spaces and related covariances.

Let $H^{2,s}$ be the Sobolev space obtained by completion of the smooth vector fields with respect to the norm $\langle(-\Delta + m^2)^s V, V\rangle_2$ (with $\langle V, V\rangle_2 = \int \|V(x)\|^2\, dx$), where m is positive. Note that the definition of $H^{2,s}$ does not depend on m.

Let a and b be nonnegative reals. Take $a_l = \frac{a}{(l-1)^{\alpha+1}}$ and $b_l = \frac{b}{(l-1)^{\alpha+1}}$ for $l \geq 1$ and $a_1 = b_1 = 0$. For $\alpha > 0$, set $G(\varphi) = \sum_{l\geq 2} \frac{1}{(l-1)^{\alpha+1}}\gamma_l(\cos\varphi)$. The function G is well defined on $[0, \pi]$ as $|\gamma_l| \leq 1$.

Let F_d and F_δ be real functions such that for all $l \geq 2$

$$(l-1)^{\alpha+1} \dim \mathcal{G}_l \times F_d(-l(l+d-1)) = d \tag{9.9}$$

$$(l-1)^{\alpha+1} \dim \mathcal{D}_l \times F_\delta(-(l+1)(l+d-2)) = d \tag{9.10}$$

and $F_d(-d) = F_\delta(-2(d-1)) = 0$. Note that when $d = 2$, $F_d = F_\delta$.

Let Π be the orthonormal projection on the space of the L^2–gradient vector fields.

Proposition 9.1 *The covariance function defined by the sequences (a_l) and (b_l) is given by (9.1) with the functions*

$$\alpha(\cos\varphi) = aG(\varphi) + b\left(\cos\varphi\, G(\varphi) + \frac{\sin\varphi}{d-1} \times G'(\varphi)\right), \tag{9.11}$$

$$\beta(\cos\varphi) = -\frac{a}{\sin\varphi}G'(\varphi) + b\left(-G(\varphi) + \frac{\cos\varphi}{(d-1)\sin\varphi} \times G'(\varphi)\right). \tag{9.12}$$

When a and b are positive, the associated self–reproducing space is $H^{2,\frac{\alpha+d}{2}}$ equipped with a different (but equivalent) norm, namely

$$\|V\|_H^2 = \frac{1}{a}\|\Pi V\|_d^2 + \frac{1}{b}\|(I-\Pi)V\|_\delta^2,$$

where $\|V\|_d^2 = \langle F_d(\Delta)^{-1}V, V\rangle_2$ and $\|V\|_\delta^2 = \langle F_\delta(\Delta)^{-1}V, V\rangle_2$.

Proof. It is not difficult to see that the norm $\|.\|_H$ given in the proposition is the norm on the self-reproducing space associated to C.

Now since (see [12])

$$\begin{aligned}\dim \mathcal{G}_l &= \frac{(d+l-3)!}{(d-1)!(l-1)!}(d+2l-3)(d+1),\\ \dim \mathcal{D}_l &= \frac{(d+l-3)!}{(d-1)!(l-1)!}(d+2l-3)\frac{d(d+1)}{2},\end{aligned}$$

for $\lambda \to \infty$, $\lambda^{\frac{\alpha+d}{2}}F_d(\lambda) = O(1)$ and $\lambda^{\frac{\alpha+d}{2}}F_\delta(\lambda) = O(1)$. This implies that $\|.\|_H$ and the norm used to define $H^{2,\frac{\alpha+d}{2}}$ are equivalent (when a and b are positive). And we get that the self–reproducing space associated to C is $H^{2,\frac{\alpha+d}{2}}$. $\square$

Remark 9.2 *If a or b vanishes, the self–reproducing space is $H^{2,\frac{\alpha+d}{2}}$ restricted to divergent free vector fields or gradient vector fields.*

9.2 Phase transitions for the Sobolev statistical solution.

Let P_t be the semigroup of the Brownian motion of variance $(a+b)G(0)$ and S_t be the statistical solution associated to P_t and C.

Let (X_t, Y_t) be the two–point motion. Let $\psi_t = d(X_t, Y_t)$. Since $h(x,y) = d^2(x,y)$ is a C^2-function, h belongs to $\mathcal{D}(A^{(2)})$ and since X_t and Y_t are solutions of an S.D.E. like (3.3), ψ_t^2 is a diffusion on $[0,\pi^2]$ and is solution of an S.D.E. ψ_t is also a diffusion on $[0,\pi]$ (note that $d(x,y)$ a priori does not belong to $\mathcal{D}(A^{(2)})$). This diffusion is eventually reflected (or absorbed) in 0 and π. Its generator is $L = \sigma^2(\varphi)\frac{d^2}{d\varphi^2} + b(\varphi)\frac{d}{d\varphi}$ (see Raimond [25]), with

$$\sigma^2(\varphi) = \alpha(1) - \alpha(\cos\varphi)\cos\varphi + \beta(\cos\varphi)\sin^2\varphi, \tag{9.13}$$

$$b(\varphi) = \frac{(d-1)}{\sin\varphi}(\alpha(1)\cos\varphi - \alpha(\cos\varphi)). \tag{9.14}$$

The generator of ψ_t^2 is $L' = \widetilde{\sigma}^2(x)\frac{d^2}{dx^2} + \widetilde{b}(x)\frac{d}{dx}$, with

$$\widetilde{\sigma}^2(x) = 4x\sigma^2(\sqrt{x}), \tag{9.15}$$

$$\widetilde{b}(x) = 2\sigma^2(\sqrt{x}) + 2\sqrt{x}b(\sqrt{x}). \tag{9.16}$$

Lemma 9.3 *If $\alpha > 2$, the statistical solution is a flow of maps.*

Proof. We have $A^{(2)}d^2(x,y) = 2\sigma^2(d(x,y)) + 2b(d(x,y))d(x,y)$. When $\alpha > 2$, then G is C^2, this implies that α is C^2 and β is continuous. Hence equation (8.1) can be checked. □

Suppose $a + b > 0$ and let $\eta = \frac{b}{a+b}$.

Theorem 9.4 *For any $\alpha \in]0,2[$,*

- *For $d = 2$ or 3 and $\eta < 1 - \frac{d}{\alpha^2}$, the statistical solution is a coalescent flow of maps.*
- *For $d = 2$ or 3 and $1 - \frac{d}{\alpha^2} < \eta < \frac{1}{2} - \frac{(d-2)}{2\alpha}$, the statistical solution is diffusive with hitting.*
- *For $d = 2$ or 3 and $\eta > \frac{1}{2} - \frac{(d-2)}{2\alpha}$ or for $d \geq 4$, the statistical solution is diffusive without hitting.*

Remark. The same phase transition appears in the $\mathbb{R}^d$ case (see theorem 10.1 below). It has been independently observed, in the context of the advection of a passive scalar, by Gawedzky and Vergassola [14].

Lemma 9.5 *For $\alpha \in]0,2[$, we have*

- *G is differentiable on $]0,\pi[$.*
- $\lim_{\varphi\to 0+} \frac{G(0)-G(\varphi)}{\varphi^\alpha} = \int_0^\pi \int_0^\infty \frac{\cos^2\theta}{t^2+\cos^2\theta} t^{\alpha-1} \sin^d\theta \ \frac{dt\ d\theta}{\Gamma(\alpha+1)c_d} = KG(0).$
- $\lim_{\varphi\to 0+} \frac{G'(\varphi)}{\varphi^{\alpha-1}} = -\alpha \int_0^\pi \int_0^\infty \frac{\cos^2\theta}{t^2+\cos^2\theta} t^{\alpha-1} \sin^d\theta \ \frac{dt\ d\theta}{\Gamma(\alpha+1)c_d} = -\alpha KG(0).$

The proof of lemma 9.5 is in appendix A. From this lemma, we get as φ goes to 0

$$\alpha(\cos\varphi) = (a+b)G(0) - \left(a + \left(1+\frac{\alpha}{d-1}\right) b\right) KG(0)\varphi^\alpha + o(\varphi^\alpha) \tag{9.17}$$

$$\beta(\cos\varphi) = \alpha\left(a - \frac{b}{d-1}\right) KG(0)\varphi^{\alpha-2} + o(\varphi^{\alpha-2}). \tag{9.18}$$

Hence,

$$\sigma^2(\varphi) = (a+b)KG(0)(\alpha+1-\alpha\eta)\varphi^\alpha(1+o(1)) \tag{9.19}$$

$$b(\varphi) = (a+b)KG(0)(d-1+\alpha\eta)\varphi^{\alpha-1}(1+o(1)). \tag{9.20}$$

In order to prove theorem 9.4, we need to study the two–point motion. Because of isotropy, it is enough to study the diffusion ψ_t. This diffusion satisfies an S.D.E. until it exits $]0,\pi[$.

Let s be the scale function of the diffusion ψ_t,

$$s(x) = \int_{x_0}^x \exp\left[-\int_{x_0}^y \frac{b(\varphi)}{\sigma^2(\varphi)}\ d\varphi\right]\ dy, \text{ with } (x_0,x) \in]0,\pi[^2.$$

Let $x \in \{0,\pi\}$ and $T_x = \inf\{t>0;\ \psi_t = x\}$. Using Breiman's terminology (see [4] p.368-369), x is an open boundary point if $T_x = \infty$ and is a closed boundary point if $T_x < \infty$. Note that x is an open boundary point if $|s(x)| = \infty$.

Firstly we are going to show that π is an open boundary point. Then :

- When $d = 2$ or 3 and $\eta < 1 - \frac{d}{\alpha^2}$, we prove that 0 is an exit boundary point (this implies that the statistical solution is a coalescent flow of maps).
- When $d = 2$ or 3 and $1 - \frac{d}{\alpha^2} < \eta < \frac{1}{2} - \frac{(d-2)}{2\alpha}$, we prove that 0 is an instantaneously reflecting regular boundary point (this implies that the statistical solution is diffusive with hitting).

- When $\eta > \frac{1}{2} - \frac{(d-2)}{2\alpha}$, we prove that 0 is an open entrance boundary point (this implies that the statistical solution is diffusive without hitting).

Lemma 9.6 *π is an open boundary point.*

Proof. It is easy to check that $s(\pi-) = \infty$ using the fact that $\alpha(1) + \alpha(-1) > 0$:

$$\alpha(1) + \alpha(-1) = (a+b)G(0) + (a-b)G(\pi) > (a+b)G(\pi) + (a-b)G(\pi) \geq 0. \quad \square$$

Since π is an open boundary point, we now study the behavior of ψ_t at and near 0.

Lemma 9.7 *If $\eta > \frac{1}{2} - \frac{(d-2)}{2\alpha}$, $s(0+) = -\infty$ and if $\eta < \frac{1}{2} - \frac{(d-2)}{2\alpha}$, $s(0+) > -\infty$.*

Proof. Let us note $\mu = \frac{d-1+\alpha\eta}{\alpha+1-\alpha\eta}$. Then, we have that $\frac{b(\varphi)}{\sigma^2(\varphi)} = \frac{\mu}{\varphi}(1+o(1))$ and for any positive ε there exist positive constants C_1 and C_2 such that for $y \leq x_0$,

$$C_1 y^{-\mu+\varepsilon} \leq \exp\left[-\int_{x_0}^{y} \frac{b(\varphi)}{\sigma^2(\varphi)}\, d\varphi\right] \leq C_2 y^{-\mu-\varepsilon}. \tag{9.21}$$

From this, we see that $s(0+) = -\infty$ if $\mu > 1$ (or if $\eta > \frac{1}{2} - \frac{(d-2)}{2\alpha}$) and $s(0+)$ is finite if $\mu < 1$ (or if $\eta < \frac{1}{2} - \frac{(d-2)}{2\alpha}$). $\square$

Lemma 9.6 and 9.7 implies that (see theorem VI-3.1 in [16]) if $\eta < \frac{1}{2} - \frac{(d-2)}{2\alpha}$ we have $T_0 < \infty$, $T_\pi = \infty$ a.s. and if $\eta > \frac{1}{2} - \frac{(d-2)}{2\alpha}$, 0 is an open boundary point and we have $\liminf \psi_t = 0$ and $\limsup \psi_t = \pi$ a.s (ψ_t is recurrent).

Remark 9.8 *When $d \geq 4$ and $\alpha \in]0,2[$, $\frac{1}{2} - \frac{(d-2)}{2\alpha} < 0$. This implies that* $\liminf \psi_t = 0$ *and* $\limsup \psi_t = \pi$ *a.s.*

Since π is an open boundary point, $\psi_t \in [0,\pi[$ for every positive t and ψ_t^2 is a solution of the S.D.E.

$$d\psi_t^2 = \sqrt{2}\widetilde{\sigma}(\psi_t^2)\, dB_t + \widetilde{b}(\psi_t^2)\, dt. \tag{9.22}$$

Note that 0 is a solution of this S.D.E. (since $\widetilde{\sigma}(0) = \widetilde{b}(0) = 0$). The solutions of this S.D.E. might be not unique.

Let $m(dx)$ be the speed measure of the diffusion :

$$m(dx) = 1_{]0,\pi[}(x) \exp\left[\int_{x_0}^{x} \frac{b(\varphi)}{\sigma^2(\varphi)}\, d\varphi\right]\, \frac{dx}{\sigma^2(x)} + m(\{0\})\delta_0 = g(x)\, dx + m(\{0\})\delta_0,$$

with $x_0 \in]0,\pi[$.

Lemma 9.9 *If $\eta > \frac{1}{2} - \frac{(d-2)}{2\alpha}$, 0 is an entrance open boundary point.*

Proof. When $\eta > \frac{1}{2} - \frac{(d-2)}{2\alpha}$, 0 is an open boundary point. From Proposition 16.45 in [4], 0 is an entrance boundary point if and only if $\int_{0+} |s(x)|m(dx) < \infty$. For any positive ε, there exists a positive constant D such that, for any $x \in]0, x_0[$,

$$|s(x)g(x)| \leq D\ x^{(\frac{\varepsilon}{c}-\alpha-\varepsilon)\wedge 0} x^{-\frac{\varepsilon}{c}-\varepsilon+1} \leq D\ x^{1-\alpha-2\varepsilon}.$$

This shows that $\int_{0+} s(x)m(dx) < \infty$ (choose ε such that $2\varepsilon \leq 2-\alpha$). □

This lemma implies that when $\eta > \frac{1}{2} - \frac{(d-2)}{2\alpha}$, there exists a positive t, a positive α and $p \in]0,1[$ such that for any $x \in]0,\pi[$, $P_x[\psi_t > \alpha] > p$. Proposition 6.8 implies that S_t is not a flow of maps and since 0 is open, S_t is diffusive without hitting.

Let now $d \in \{2,3\}$ (when $d \geq 4$ we always have $\eta > \frac{1}{2} - \frac{(d-2)}{2\alpha}$).

Lemma 9.10 *If $\eta < \frac{1}{2} - \frac{(d-2)}{2\alpha}$, 0 is a closed boundary point.*

Proof. From Proposition 16.43 p.366 in [4], T_0 is finite or the boundary point 0 is closed if and only if for any $b \in]0,\pi[$, $\int_0^b |s(x) - s(0)|m(dx)$ is finite.

We have $|s(x)-s(0)|g(x) \sim \int_0^x \exp\left[-\int_{x_0}^y \frac{b(\varphi)}{\sigma^2(\varphi)}\ d\varphi\right] \times \frac{1}{\sigma^2(x)} \exp\left[\int_{x_0}^y \frac{b(\varphi)}{\sigma^2(\varphi)}\ d\varphi\right]\ dy$. Hence $|s(x)-s(0)|g(x) = O(x^{1-\alpha})$. This implies that $\int_0^b |s(x)-s(0)|m(dx)$ is finite. This proves that T_0 is finite a.s. □

Lemma 9.11 *If $\eta < 1 - \frac{d}{\alpha^2}$, 0 is an exit boundary point.*

Proof. In [4], 0 is an exit boundary point if and only if $m(]0,x[) = \infty$ for all $x \in]0,\pi[$. This is the case if $\mu - \alpha < -1$ (or if $\eta < 1 - \frac{d}{\alpha^2}$). Note that for $d = 2$ or 3 and $\alpha \in]0,2[$, $1 - \frac{d}{\alpha^2} < \frac{1}{2} - \frac{(d-2)}{2\alpha}$. □

Lemma 9.11 implies that when $\eta < 1 - \frac{d}{\alpha^2}$, the diffusion ψ_t is absorbed at 0, and for any positive r,

$$\lim_{d(x,y)\to 0} P^{(2)}_{(x,y)}[d(X_t,Y_t) > r] = \lim_{\varphi\to 0} P_\varphi[\psi_t > r] = 0.$$

Now, applying proposition 6.7, we prove that the statistical solution is a flow of maps and this is a coalescent flow of maps (since 0 is an exit boundary point).

Lemma 9.12 *If $\eta \in \left]1-\frac{d}{\alpha^2}, \frac{1}{2}-\frac{(d-2)}{2\alpha}\right[$, 0 is a regular boundary point.*

Proof. In [4], we see that 0 is regular if $m(]0,x[) < \infty$ for all $x \in]0,\pi[$, which is the case when $\eta \in \left]1-\frac{d}{\alpha^2}, \frac{1}{2}-\frac{(d-2)}{2\alpha}\right[$. $\square$

When $\eta \in \left]1-\frac{d}{\alpha^2}, \frac{1}{2}-\frac{(d-2)}{2\alpha}\right[$, the two–point motion hits the diagonal. But there is no uniqueness of the solution of the S.D.E. satisfied by ψ_t since 0 might be absorbing or (slowly or instantaneously) reflecting. In order to finish the proof of theorem 9.4, we are going to prove that 0 is instantaneously reflecting.

To prove this, for $\varepsilon \in]0,1[$, let us introduce the covariance $C_\varepsilon = (1-\varepsilon)^2 C$ (then, if W_t is the cylindrical Brownian motion associated to C, $(1-\varepsilon)W_t$ is the cylindrical Brownian motion associated to C_ε) and S_t^ε be the statistical solution associated to P_t and C_ε.

For $f \in L^2(dx)$, $S_t^\varepsilon f = \sum_{n\geq 0} J_t^{n,\varepsilon} f$, where $J_t^{n,\varepsilon} f$ is the nth chaos in the chaos expansion of $S_t^\varepsilon f$ [2]. It is easy to see that $J_t^{n,\varepsilon} f = (1-\varepsilon)^n J_t^n f$, where $J_t^n f$ is the nth chaos in the chaos expansion of $S_t f$, hence

$$E[(S_t^\varepsilon f - S_t f)^2] = \sum_{n\geq 1} (1-(1-\varepsilon)^{2n}) E[(J_t^n f)^2]. \tag{9.23}$$

Hence it is clear that the $L^2(P)$-limit as ε goes to 0 of $S_t^\varepsilon f$ is $S_t f$.

Let $(X_t^\varepsilon, Y_t^\varepsilon)$ be the Markov process associated to $P_t^{(2),\varepsilon} = E[S_t^{\varepsilon\,\otimes 2}]$ and $\psi_t^\varepsilon = d(X_t^\varepsilon, Y_t^\varepsilon)$. ψ_t^ε is a diffusion with generator L_ε. It is easy to see that $L_\varepsilon = (1-(1-\varepsilon)^2)L_1 + (1-\varepsilon)^2 L$ (note that $A_\varepsilon^{(2)} = A\otimes I + I\otimes A + (1-\varepsilon)^2 C = A_1^{(2)} + (1-\varepsilon)^2(A^{(2)} - A_1^{(2)})$), and $L_\varepsilon = \sigma_\varepsilon^2(\varphi)\frac{d^2}{d\varphi^2} + b_\varepsilon(\varphi)\frac{d}{d\varphi}$, with

$$\begin{aligned} \sigma_\varepsilon^2(\varphi) &= (1-(1-\varepsilon)^2)\sigma_1^2(\varphi) + (1-\varepsilon)^2\sigma^2(\varphi), & (9.24)\\ b_\varepsilon(\varphi) &= (1-(1-\varepsilon)^2) b_1(\varphi) + (1-\varepsilon)^2 b(\varphi). & (9.25) \end{aligned}$$

Let us remark that L_1 is the generator of the diffusion distance between two independent Brownian motions on S^d. Note that as φ goes to 0,

$$\sigma_1^2(\varphi) \sim \sigma_1^2(0) = 2(a+b)KG(0) \qquad \text{and} \qquad b_1(\varphi) \sim \frac{2(d-1)}{\varphi}(a+b)KG(0) \tag{9.26}$$

[2]Note that $S_t^\varepsilon = Q_{\log(1-\varepsilon)} S_t$, where Q_α is the Ornstein-Uhlenbeck operator on the Wiener space (used in Malliavin calculus : see [24]).

and $\sigma_\varepsilon^2(\varphi) = (1-(1-\varepsilon)^2)\sigma_1^2(\varphi)(1+O(\varphi^\alpha))$ and $b_\varepsilon(\varphi) = (1-(1-\varepsilon)^2)b_1(\varphi)(1+O(\varphi^\alpha)$. Studying the scale function s_ε of ψ_t^ε, we get that $s_\varepsilon(0+) = s_1(0+) = -\infty$ (as two independent Brownian motions cannot meet each other on S^d). We still have $s_\varepsilon(\pi-) = \infty$. Hence $\psi_t^\varepsilon \in]0,\pi[$ for all positive t.

Let m_ε be the speed measure of ψ_t^ε. Let $g_\varepsilon(x) = m_\varepsilon(dx)/dx$. As $m_\varepsilon(]0,\pi[) < \infty$, m_ε is an invariant finite measure for the diffusion ψ_t^ε. As $\lim_{\varepsilon\to 0}\sigma_\varepsilon^2 = \sigma^2$ and $\lim_{\varepsilon\to 0} b_\varepsilon = b$, we get that $\lim_{\varepsilon\to 0} g_\varepsilon(x) = g(x)$. Let us note $\varepsilon' = 1-(1-\varepsilon)^2$ and let

$$f(\varepsilon',\varphi) = \frac{\varepsilon' b_1(\varphi) + (1-\varepsilon')b(\varphi)}{\varepsilon'\sigma_1^2(\varphi) + (1-\varepsilon')\sigma^2(\varphi)}. \tag{9.27}$$

This function increases with ε' if $\frac{b_1(\varphi)}{\sigma_1^2(\varphi)} \geq \frac{b(\varphi)}{\sigma^2(\varphi)}$. As $\frac{b_1(\varphi)}{\sigma_1^2(\varphi)} - \frac{b(\varphi)}{\sigma^2(\varphi)} \sim (d-1-\mu)\frac{1}{\varphi}$ as φ goes to 0 and as $(d-1-\mu)$ is positive, there exists φ_0 such that for any $\varphi < \varphi_0$, $f(\varepsilon',\varphi) \geq \frac{b(\varphi)}{\sigma^2(\varphi)} = f(0,\varphi)$ and for $\varepsilon' < 1/2$,

$$g_\varepsilon(x) \leq \frac{2}{\sigma^2(x)} \exp\left(-\int_x^{\varphi_0} \frac{b(\varphi)}{\sigma^2(\varphi)}\, d\varphi\right) \times C_{\varphi_0},$$

where $C_{\varphi_0} = \sup_{\varepsilon\in[0,1]} \exp\left(\int_{\varphi_0}^{x_0} f(\varepsilon',\varphi)\, d\varphi\right) < \infty$. The Lebesgue's dominated convergence theorem implies that g_ε converges in $L^1([0,\pi])$ towards g.

Let f and g be continuous functions, then $E[f(X_t^\varepsilon)g(Y_t^\varepsilon)] = E[S_t^\varepsilon f(x) S_t^\varepsilon g(y)]$. Since $S_t^\varepsilon f$ and $S_t^\varepsilon g$ converge respectively towards $S_t f$ and $S_t g$ when ε goes to 0 in $L^2(P)$, we get that $(X_t^\varepsilon, Y_t^\varepsilon)$ converges in distribution towards (X_t, Y_t) when ε goes to 0. This also implies that ψ_t^ε converges in distribution towards ψ_t when ε goes to 0.

Since m_ε is an invariant measure, for any continuous function f on $[0,\pi]$, we have

$$\int E[f(\psi_t^\varepsilon)|\psi_0^\varepsilon = x]\ m_\varepsilon(dx) = \int f\ dm_\varepsilon. \tag{9.28}$$

Since

$$\begin{aligned} &\left|\int E[f(\psi_t^\varepsilon)|\psi_0^\varepsilon = x]\ m_\varepsilon(dx) - \int E[f(\psi_t)|\psi_0 = x]\ g(x)dx\right| \leq \\ &\quad \leq \|f\|_\infty \int_0^\pi |g_\varepsilon(x) - g(x)|\ dx + \left|\int_0^\pi \left(E[f(\psi_t^\varepsilon)|\psi_0^\varepsilon = x] - E[f(\psi_t)|\psi_0 = x]\right)\ g(x)dx\right|, \end{aligned}$$

we get that (because g_ε converges in $L^1([0,\pi])$ towards g and ψ_t^ε converges in distribution toward ψ_t.)

$$\int E[f(\psi_t)|\psi_0 = x]\ m(dx) = \lim_{\varepsilon\to 0}\int E[f(\psi_t^\varepsilon)|\psi_0^\varepsilon = x]\ m_\varepsilon(dx) = \lim_{\varepsilon\to 0}\int f\ dm_\varepsilon = \int f\ dm.$$

This implies that $g(x)dx$ is an invariant measure for ψ_t and $m(dx) = g(x)dx$. Since $m(]0,x[) < \infty$ for all $x \in]0,\pi[$, the diffusion ψ_t is not absorbed in 0 and is reflected in 0.

In this case, 0 is a closed regular boundary point. This point is instantaneously reflecting since $m(\{0\}) = 0$. This implies the existence of a positive t, a positive r and $p \in]0,1]$ such that for any $x \in]0,\pi[$, $P_x[\psi_t \geq r] \geq p$. Then, applying proposition 6.8, the statistical solution is not a flow of maps. This finishes the proof of theorem 9.4. □

For $\alpha > 2$, the statistical solution is an isotropic Brownian flow of diffeomorphisms. In Raimond [25], the Lyapunov exponents of this flow are computed. The sign of the first Lyapunov exponent $\lambda_1(\alpha,d)$ describes the stability of the flow. It is unstable if $\lambda_1 \geq 0$ and stable if $\lambda_1 < 0$. The computation of $\lambda_1(\alpha,d)$ gives

$$\lambda_1 = \frac{(d-4)a+db}{d+2}\zeta(\alpha-1) + \left(\frac{d-1}{d+2}\right)[(d-4)a+db]\zeta(\alpha) - d\left(\frac{2(d-1)a+db}{d+2}\right)\zeta(\alpha+1), \tag{9.29}$$

where $\zeta(\alpha) = \sum_{l\geq 1}\frac{1}{l^\alpha}$ is the zeta function. Therefore, we have $\lambda_1(\alpha,d) = 0$ if and only if

$$\eta = \eta(\alpha,d) = \frac{-(d-4)\zeta(\alpha-1) - (d-1)(d-4)\zeta(\alpha) + 2d(d-1)\zeta(\alpha+1)}{4\zeta(\alpha-1) + 4(d-1)\zeta(\alpha) + d(d-2)\zeta(\alpha+1)}. \tag{9.30}$$

It is easy to see that for fixed η, $\lim_{\alpha\to 2+}\lambda_1(\alpha,d) = +\infty$ if $d \geq 4$ or if $\eta > \frac{1}{2} - \frac{d-2}{4} = \frac{4-d}{4}$ and that $\lim_{\alpha\to 2+}\lambda_1(\alpha,d) = -\infty$ if $\eta < \frac{4-d}{4}$. Remark that $\lim_{\alpha\to 2-} 1 - \frac{d}{\alpha^2} = \lim_{\alpha\to 2-}\frac{1}{2} - \frac{(d-2)}{2\alpha} = \frac{4-d}{4}$. This shows that coalescence appears when λ_1 goes to $-\infty$ and splitting appears when λ_1 goes to $+\infty$.

The results of this section is given by phase diagrams in appendix B.

10 Isotropic statistical solution on $\mathbb{R}^d$.

10.1 Stationary and isotropic covariance functions on $\mathbb{R}^d$.

On $\mathbb{R}^d$ with $d \geq 2$, the stationary isotropic covariance function C are (see Le Jan [20]) such that $C^{ij}(x,y) = C^{ij}(x-y)$, for $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$, with

$$C^{ij}(z) = \delta^{ij}B_N(\|z\|) + \frac{z^iz^j}{\|z\|^2}(B_L(\|z\|) - B_N(\|z\|)), \tag{10.1}$$

with

$$B_L(r) = \int\int \cos(\rho u_1 r) u_1^2 \omega(du)(F_L(d\rho) - F_N(d\rho)) + \int\int \cos(\rho u_1 r)\omega(du) F_N(d\rho), \quad (10.2)$$

$$B_N(r) = \int\int \cos(\rho u_1 r) u_2^2 \omega(du)(F_L(d\rho) - F_N(d\rho)) + \int\int \cos(\rho u_1 r)\omega(du) F_N(d\rho), \quad (10.3)$$

F_L and F_N being finite positive measures on $\mathbb{R}^+$. $\omega(du)$ is the normalized Lebesgue measure on S^{d-1}. F_L and F_N represent respectively the gradient part and the zero divergence part of the associated Gaussian vector field.

For α and m positive reals, let $F(d\rho) = \frac{\rho^{d-1}}{(\rho^2+m^2)^{\frac{d+\alpha}{2}}} d\rho$, $F_L(d\rho) = aF(d\rho)$ and $F_N(d\rho) = \frac{b}{d-1}F(d\rho)$, where a and b are nonnegative. In the Fourier representation, (c is a positive constant)

$$\hat{C}^{ij}(k) = c(\|k\|^2 + m^2)^{-\frac{d+\alpha}{2}} \left(a\frac{k^i k^j}{\|k\|^2} + \frac{b}{d-1}\left(\delta_{ij} - \frac{k^i k^j}{\|k\|^2}\right)\right). \quad (10.4)$$

Notice that in the Fourier representation, the Laplace operator on vector fields is given by the multiplication by $-\|k\|^2$ and the projection π on gradient vector fields (in the L^2 space) by $\frac{k^i k^j}{\|k\|^2}$. (i.e if V is a vector field and $\hat{V}^i(k)$ its Fourier transform, $(\hat{\pi V})^i(k) = \sum_j \frac{k^i k^j}{\|k\|^2}\hat{V}^j(k)$.)

Therefore, given a L^2 vector field, $U^j(y) = \int \sum_i C^{ij}(x-y)V^i(x)\, dx$ can be expressed as $c(-\Delta + m^2)^{-\frac{d+\alpha}{2}}(a\pi V + \frac{b}{d-1}(I-\pi)V)$. Since $\langle U, U\rangle_H = \langle U, V\rangle_2 = \int \langle U(x), V(x)\rangle\, dx$, the self–reproducing space appears to be the L^2-Sobolev space of order $s = \frac{d+\alpha}{2}$ (defined the same way as in secion 9.1) equipped with the norm

$$\|V\|^2 = \frac{1}{a}\|\pi V\|_s^2 + \frac{d-1}{b}\|(I-\pi)V\|_s^2,$$

where

$$\|V\|_s^2 = \frac{1}{c}\langle(-\Delta + m^2)^s V, V\rangle_2.$$

Note that if a or b vanishes, the self–reproducing space is $H^{2,\frac{\alpha+d}{2}}$ restricted to divergence free vector fields or gradient vector fields.

10.2 Phase transitions for the Sobolev statistical solution.

Let P_t be the semigroup of a Brownian motion on $\mathbb{R}^d$ with variance $(a+b)F(\mathbb{R}^+)$. Let S_t be the statistical solution associated to P_t and C. If $\alpha > 2$, C is C^2. Hence equation (8.2) is satisfied and the statistical solution S_t is a flow of maps.

Suppose $a+b>0$ and let $\eta = \frac{b}{a+b}$. Then we have the theorem.

Theorem 10.1 *For any $\alpha \in]0,2[$,*

- *For $d=2$ or 3 and $\eta < 1-\frac{d}{\alpha^2}$, the statistical solution is a coalescent flow of maps.*
- *For $d=2$ or 3 and $1-\frac{d}{\alpha^2} < \eta < \frac{1}{2} - \frac{(d-2)}{2\alpha}$, the statistical solution is diffusive with hitting.*
- *For $d=2$ or 3 and $\eta > \frac{1}{2} - \frac{(d-2)}{2\alpha}$ or for $d \geq 4$, the statistical solution is diffusive without hitting.*

Remark. The results of this theorem are exactly the same as for the sphere.

Proof. Let us study the two–point motion (X_t, Y_t) starting from (x,y) (with $x \neq y$). Then $r_t = d(X_t, Y_t)$ is a diffusion in $\mathbb{R}^+$ (eventually reflected in 0), with generator $L = \sigma^2(r)\frac{d^2}{dr^2} + b(r)\frac{d}{dr}$ (see Le Jan [20]), with

$$\begin{aligned} \sigma^2(r) &= B - B_L(r), & (10.5)\\ b(r) &= (d-1)\frac{B - B_N(r)}{r}, & (10.6) \end{aligned}$$

where $B = B_L(0) = B_N(0) = \frac{a+b}{d}F(\mathbb{R}^+)$.

Lemma 10.2 *For $\alpha \in]0,2[$, as r goes to 0,*

i) $\iint \cos(\rho u_1 r)\omega(du)F(d\rho) = F(\mathbb{R}^+) - \alpha_1 r^\alpha + o(r^\alpha)$.

ii) $\iint \cos(\rho u_1 r)u_1^2\omega(du)F(d\rho) = \frac{F(\mathbb{R}^+)}{d} - \alpha_2 r^\alpha + o(r^\alpha)$.

iii) $\iint \cos(\rho u_1 r)u_2^2\omega(du)F(d\rho) = \frac{F(\mathbb{R}^+)}{d} - \alpha_3 r^\alpha + o(r^\alpha)$.

with $\alpha_2 = \frac{\alpha+1}{d+\alpha}\alpha_1$, $\alpha_3 = \frac{1}{d+\alpha}\alpha_1$ and

$$\alpha_1 = c_d \left(\int_0^\infty (1-\cos x)\frac{dx}{x^{\alpha+1}} \right) \left(\int_0^{\frac{\pi}{2}} (\cos\theta)^\alpha (\sin\theta)^{d-2} d\theta \right).$$

Proof. For $r > 0$, making the change of variable $x = \rho u_1 r$,

$$\begin{aligned}\iint (1-\cos(\rho u_1 r))\omega(du)F(d\rho) &= c_d \int_0^1 \int_0^\infty (1-\cos(\rho u_1 r))(1-u_1^2)^{\frac{d-2}{2}} du_1 \frac{\rho^{d-1}d\rho}{(\rho^2+m^2)^{\frac{d+\alpha}{2}}} \\ &= r^\alpha c_d \int_0^1 \left(\int_0^\infty (1-\cos x) \frac{x^{d-1}dx}{(x^2+r^2u_1^2m^2)^{\frac{d+\alpha}{2}}} \right) u_1^\alpha (1-u_1^2)^{\frac{d-2}{2}} du_1.\end{aligned}$$

As $\lim_{r\to 0} \int_0^\infty (1-\cos x) \frac{x^{d-1}}{(x^2+r^2u_1^2m^2)^{\frac{d+\alpha}{2}}} dx = \int_0^\infty (1-\cos x)\frac{dx}{x^{\alpha+1}} < \infty$, we get that

$$\lim_{r\to 0} \frac{1}{r^\alpha} \iint (1-\cos(\rho u_1 r))\omega(du)F(d\rho) = c_d \left(\int_0^\infty (1-\cos x) \frac{dx}{x^{\alpha+1}} \right) I(d-2,\alpha) = \alpha_1,$$

with $I(n,t) = \int_0^{\frac{\pi}{2}} (\cos\theta)^t (\sin\theta)^n \, d\theta = \frac{1}{2}B(\frac{n+1}{2}, \frac{t+1}{2})$ for $t \geq 0$ and $n \in \mathbb{N}$, and $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. This shows i). ii) and iii) can be obtained the same way with

$$\alpha_2 = c_d \int_0^\infty (1-\cos x)\frac{dx}{x^{\alpha+1}} I(d-2,\alpha+2)$$

and $\alpha_1 = \alpha_2 + (d-1)\alpha_3$ (note that $\int u_1^2 \omega(du) = \frac{1}{d}$). It is easy to see that for $\alpha > 0$ and $d \geq 1$,

$$I(d-2,\alpha+2) = \frac{\alpha+1}{d+\alpha} I(d-2,\alpha).$$

Therefore, $\alpha_2 = \frac{\alpha+1}{d+\alpha}\alpha_1$. With the relation $\alpha_1 = \alpha_2 + (d-1)\alpha_3$, we get that $\alpha_3 = \frac{1}{d+\alpha}\alpha_1$. □

Remark 10.3 *As z goes to 0,*

$$C^{ij}(z) = B\delta^{ij} - \frac{\alpha_1}{d-1}\left[((d-1)a + (d+\alpha-1)b)\delta^{ij} - \alpha((d-1)a - b)\frac{z^i z^j}{\|z\|^2} \right] \|z\|^\alpha (1+o(1)),$$

Let us note that the dependence on m only appears in B.

From this lemma, it is easy to see that as r goes to 0,

$$\sigma^2(r) = \frac{(a+b)\alpha_1}{d+\alpha}(\alpha+1-\alpha\eta) r^\alpha (1+o(1)), \tag{10.7}$$

$$b(r) = \frac{(a+b)\alpha_1}{d+\alpha}(d-1+\alpha\eta) r^{\alpha-1}(1+o(1)). \tag{10.8}$$

Note that we get the same behaviour of σ and b around 0 as in section 9.2.

As in section 9.2, let us study s, the scale function of the diffusion r_t.

Since $B_L(r)$ and $B_N(r)$ converge towards 0 as r goes to ∞ (as Fourier transforms of finite measures), we get that as r goes to ∞, $\log(s'(r)) \sim (1-d)\log(r)$. Therefore $s(+\infty)$ is finite if and only if $d \geq 3$.

We also see that $s(0+) = -\infty$ if $\eta > \frac{1}{2} - \frac{(d-2)}{2\alpha}$ and $s(0+)$ is finite if $\eta < \frac{1}{2} - \frac{(d-2)}{2\alpha}$.

Let m be the speed measure of the diffusion. Let us study the boundary point 0.

As $m(]0,x[) < -\infty$ for any positive x if $\eta > 1 - \frac{d}{\alpha^2}$, as in section 9.2, with a similar proof, we can prove that if $\eta \in]1 - \frac{d}{\alpha^2}, \frac{1}{2} - \frac{(d-2)}{2\alpha}[$, the diffusion r_t is instantaneously reflecting at 0. The only thing there is to change in the proof is to take the test function f in (9.28) with compact support and to remark that g_ε converges towards g in $L^1_{loc}(\mathbb{R}^+)$.

If $\eta < 1 - \frac{d}{\alpha^2}$ (note that $1 - \frac{d}{\alpha^2} \leq \frac{1}{2} - \frac{d-2}{\alpha}$), 0 is an exit boundary point and the diffusion is absorbed by 0.

Therefore, we get that

- If $d \geq 3$ and $\eta \in]1 - \frac{d}{\alpha^2}, \frac{1}{2} - \frac{(d-2)}{2\alpha}[$, r_t is instantaneously reflecting at 0 and is transient. In this case, as in section 9.2, $(S_t)_{t\geq 0}$ is diffusive with hitting.

- If $d = 2$ and $\eta \in]1 - \frac{d}{\alpha^2}, \frac{1}{2} - \frac{(d-2)}{2\alpha}[$, r_t is instantaneously reflecting at 0 and is recurrent. In this case, as in section 9.2, $(S_t)_{t\geq 0}$ is diffusive with hitting.

- If $d \geq 3$ and $\eta < 1 - \frac{d}{\alpha^2}$, r_t is absorbed at 0 with probability $\frac{s(\infty)-s(r_0)}{s(\infty)-s(0)}$ and converges towards $+\infty$ with probability $\frac{s(r_0)-s(0)}{s(\infty)-s(0)}$. In this case, as in the section 9.2, $(S_t)_{t\geq 0}$ is a coalescent flow of maps.

- If $d = 2$ and $\eta < 1 - \frac{d}{\alpha^2}$, r_t is absorbed at 0 a.s. In this case, as in section 9.2, $(S_t)_{t\geq 0}$ is a coalescent flow of maps.

If $\eta > \frac{1}{2} - \frac{d-2}{2\alpha}$, then we have that $s(0) = -\infty$. In this case, 0 is an entrance boundary point as $\int_{0+} |s(x)| dm(x) < \infty$. r_t is recurrent if $d = 2$ and transient if $d \geq 3$. As in the section 9.2, we prove that $(S_t)_{t\geq 0}$ is diffusive without hitting. □

For $\alpha > 2$, the statistical solution is a stationary isotropic Brownian flow of diffeomorphisms. In Le Jan [20], the Lyapunov exponents of this flow are computed. The sign of the

first Lyapunov exponent $\lambda_1(\alpha, d)$ describes the stability of the flow. It is unstable if $\lambda_1 \geq 0$ and stable if $\lambda_1 < 0$. The computation of $\lambda_1(\alpha, d)$ gives (see [20])

$$\lambda_1 = \frac{1}{2(d+2)}((d-4)a + db) \int \rho^2 F(d\rho), \tag{10.9}$$

Therefore, we have $\lambda_1(\alpha, d) = 0$ if and only if $d \leq 4$ and

$$\eta = \eta(d) = \frac{4-d}{4} \tag{10.10}$$

As in the section 9.2, we see that for fixed η, $\lim_{\alpha \to 2+} \lambda_1(\alpha, d) = +\infty$ if $d \geq 4$ or if $\eta > \frac{1}{2} - \frac{d-2}{4} = \frac{4-d}{4}$ and that $\lim_{\alpha \to 2+} \lambda_1(\alpha, d) = -\infty$ if $\eta < \frac{4-d}{4}$. This shows that coalescence appears when λ_1 goes to $-\infty$ and splitting appears when λ_1 goes to $+\infty$.

Remark that $\lim_{\alpha \to 2-} 1 - \frac{d}{\alpha^2} = \lim_{\alpha \to 2-} \frac{1}{2} - \frac{(d-2)}{2\alpha} = \frac{4-d}{4}$.

The results of this section are given by phase diagrams in appendix B.

11 Reflecting flows.

Let D be an open convex domain in $\mathbb{R}^d$ with C^1 boundary ∂D. Let d be the Euclidean metric in $\mathbb{R}^d$. For any $x \in \partial D$, we denote $n(x)$ the directed inward unit normal vector to ∂D.

Let P_t be the semigroup of the Brownian motion in D reflected on ∂D. P_t is associated to the Dirichlet form $(\mathcal{E}, \mathcal{F})$, where $\mathcal{F} = H^1(D) = \{f \in L^2(D, dx),\ |\nabla f| \in L^2(D, dx)\}$ equipped with the form $\frac{1}{2}\int_D |\nabla f|^2\, dx$ (see [11], 1.3.2). Let $C(x, y)$ be a covariance function in $D \times D$ such that $C^{ij}(x, x) = \delta^{ij}$ and satisfying (8.1).

We can construct a statistical solution associated to P_t and C. Let $P_t^{(2)}$ be the semigroup of the two–point motion (X_t, Y_t). Let $P_{(.,.)}^{(2)}$ be the law of the two–point motion.

We know that X_t and Y_t are two diffusions in D reflected on ∂D. Let φ_t and ψ_t denote the local times of X_t and Y_t on ∂D.

Lemma 11.1 *For $h(x, y) = d^2(x, y)$, $P_t^{(2)} h(x, y) \leq h(x, y) e^{Ct}$.*

Proof. Let us note

$$L^{(2)} = A_x + A_y + \sum_{i,j} C^{ij}(x, y) \partial_{x_i} \partial_{x_j}.$$

From (8.1) and the Lipschitz conditions, we get that

$$L^{(2)}h(x,y) \leq C\ h(x,y).$$

Using Tanaka's formula, there exists a martingale M_t such that

$$\begin{aligned} h(X_t,Y_t) - h(x,y) &= M_t + \int_0^t L^{(2)}h(X_s,Y_s)\ ds && (11.1) \\ &+ \int_0^t \langle \nabla_x h(X_s,Y_s), n(X_s)\rangle d\varphi_s + \int_0^t \langle \nabla_y h(X_s,Y_s), n(Y_s)\rangle d\psi_s && (11.2) \end{aligned}$$

As $\nabla_x h(x,y) = 2(x-y)$, using the fact that D is convex, we get that for $x \in \partial D$

$$\langle \nabla_x h(x,y), n(x)\rangle < 0.$$

This implies that

$$h(X_t,Y_t) - h(x,y) \leq M_t + C\int_0^t h(X_s,Y_s)\ ds.$$

Taking the expectation, we get that $P_t^{(2)}h(x,y) - h(x,y) \leq C\int_0^t P_s^{(2)}h(x,y)\ ds$. Hence the lemma. □

Theorem 11.2 *The statistical solution is a flow of maps.*

Proof. This is the same proof as the proof of theorem 8.2. □

A Proof of lemma 9.5.

Take $\varphi \in]0,\pi[$. At first, we are going to prove that $I(\varphi) = \sum_{l\geq 2}\frac{1}{(l-1)^{\alpha+1}}\left|\frac{d}{d\varphi}\gamma_l(\cos\varphi)\right|$ is finite. As $\frac{1}{l^\alpha} = \int_0^\infty e^{-ls}s^{\alpha-1}\ \frac{ds}{\Gamma(\alpha)}$,

$$\begin{aligned} I(\varphi) &\leq \int_0^\pi \int_0^\infty \sum_{l\geq 1}[e^{-s}|z(\varphi,\theta)|]^l \frac{\left|\frac{d}{d\varphi}z(\varphi,\theta)\right|}{|z(\varphi,\theta)|} s^{\alpha-1}\sin^d\theta\ \frac{ds\ d\theta}{\Gamma(\alpha)c_d} && (A.1) \\ &\leq \int_0^\pi\int_0^\infty f_{\varphi,\theta}(s)\ ds\ d\theta = 2\int_0^{\frac{\pi}{2}}\int_0^\infty f_{\varphi,\theta}(s)\ ds\ d\theta, && (A.2) \end{aligned}$$

with $f_{\varphi,\theta}(s) = \frac{e^{-s}\left|\frac{d}{d\varphi}z(\varphi,\theta)\right|}{1-e^{-s}|z(\varphi,\theta)|}\frac{s^{\alpha-1}}{\Gamma(\alpha)c_d}$. It is easy to see that

$$\int_1^\infty f_{\varphi,\theta}(s)\ ds \leq \frac{1}{\Gamma(\alpha)c_d}\int_1^\infty \frac{e^{-s}s^{\alpha-1}}{(1-e^{-s})}ds < \infty. \qquad (A.3)$$

On the other hand,

$$\int_0^1 f_{\varphi,\theta}(s)\ ds \le \frac{1}{\Gamma(\alpha)c_d}\int_0^1 \frac{ds}{1-e^{-s}|z(\varphi,\theta)|} = \frac{1}{\Gamma(\alpha)c_d}F_\varphi(\theta). \tag{A.4}$$

Let $x_\varphi(\theta) = -\log|z(\varphi,\theta)|$, then $F_\varphi(\theta) = \int_{x_\varphi(\theta)}^{x_\varphi(\theta)+1}\frac{dt}{1-e^{-t}}$. As $\lim_{\theta\to 0+} x_\varphi(\theta) = 0$, we have $F_\varphi(\theta) \sim -\log x_\varphi(\theta)$ as θ goes to 0. From this, we see that $F_\varphi(\theta) = O(\log\theta)$ as θ goes to 0. This implies that $I(\varphi)$ is finite.

Now, applying the derivation under the integral theorem, we prove that G is differentiable on $]0,\pi[$ and that for $\varphi\in]0,\pi[$,

$$\begin{aligned}
G'(\varphi) &= \sum_{l\ge 1}\int_0^\pi \frac{[z(\varphi,\theta)]^{l-1}\frac{d}{d\varphi}z(\varphi,\theta)}{l^\alpha}\sin^d\theta\ \frac{d\theta}{c_d} && \text{(A.5)}\\
&= \int_0^\pi\int_0^\infty \sum_{l\ge 1}[e^{-s}z(\varphi,\theta)]^l\frac{\frac{d}{d\varphi}z(\varphi,\theta)}{z(\varphi,\theta)}s^{\alpha-1}\sin^d\theta\ \frac{ds\ d\theta}{\Gamma(\alpha)c_d} && \text{(A.6)}\\
&= \int_0^\pi\int_0^\infty \frac{e^{-s}\frac{d}{d\varphi}z(\varphi,\theta)}{1-e^{-s}z(\varphi,\theta)}s^{\alpha-1}\sin^d\theta\ \frac{ds\ d\theta}{\Gamma(\alpha)c_d}. && \text{(A.7)}
\end{aligned}$$

As $z(\varphi,\pi-\theta) = \overline{z(\varphi,\theta)}$,

$$G'(\varphi) = -\int_0^{\frac{\pi}{2}}\int_0^\infty \frac{a(s,\varphi)-\sin^2\theta}{b(s,\varphi)+\cos^2\theta}\times\frac{\cos\varphi}{\sin\varphi}s^{\alpha-1}\sin^d\theta\ \frac{2ds\ d\theta}{\Gamma(\alpha)c_d},$$

with $a(s,\varphi) = \frac{1}{e^{-s}\cos\varphi}$ and $b(s,\varphi) = \frac{(1-e^{-s}\cos\varphi)^2}{e^{-2s}\sin^2\varphi}$. Changing of variables ($s = t\varphi$),

$$\begin{aligned}
-\frac{G'(\varphi)}{\varphi^{\alpha-1}} &= \int_0^{\frac{\pi}{2}}\int_0^\infty \frac{a(t\varphi,\varphi)-\sin^2\theta}{b(t\varphi,\varphi)+\cos^2\theta}\times\frac{\varphi\cos\varphi}{\sin\varphi}t^{\alpha-1}\sin^d\theta\ \frac{2dt\ d\theta}{\Gamma(\alpha)c_d} && \text{(A.8)}\\
&= \int_0^{\frac{\pi}{2}}\int_0^\infty I(t,\varphi,\theta)\ dt\ d\theta. && \text{(A.9)}
\end{aligned}$$

Let $\varepsilon > 0$, there exists a positive constant C_ε such that for any $t\in[0,\varepsilon]$,

$$0 \le I(t,\varphi,\theta) \le C_\varepsilon t^{\alpha-1}. \tag{A.10}$$

Remark also that

$$I(t,\varphi,\theta) \le C_{d,\alpha}\frac{t^2\varphi^2e^{-t\varphi}}{(1-e^{-t\varphi})^2}\times t^{\alpha-3}, \tag{A.11}$$

where $C_{d,\alpha}$ is a positive constant. Let $C = C_{d,\alpha}\sup_{x>0}\frac{x^2e^{-x}}{(1-e^{-x})^2} < \infty$, then, for any positive t

$$0 \le I(t,\varphi,\theta) \le C\ t^{\alpha-3}. \tag{A.12}$$

As $F(t) = C_\varepsilon t^{\alpha-1} 1_{0<t\leq\varepsilon} + C\, t^{\alpha-3} 1_{t>\varepsilon}$ belongs to $L^1(d\theta \otimes dt)$ for $\alpha \in]0,2[$, $\lim_{\varphi\to 0} a(t\varphi,\varphi) = 1$ and $\lim_{\varphi\to 0} b(t\varphi,\varphi) = t^2$, by the Lebesgue dominated convergence theorem,

$$\lim_{\varphi\to 0} \frac{G'(\varphi)}{\varphi^{\alpha-1}} = -\int_0^{\frac{\pi}{2}} \int_0^\infty \frac{\cos^2\theta}{t^2+\cos^2\theta} t^{\alpha-1} \sin^d\theta \, \frac{2d\theta\, dt}{c_d \Gamma(\alpha)} = -\alpha K. \tag{A.13}$$

We have proved the second limit. The first limit is easy to obtain as

$$\begin{aligned} G(0) - G(\varphi) &= -\int_0^\varphi G'(x)\, dx \\ &= -K\, \varphi^\alpha + o(\varphi^\alpha). \end{aligned}$$

This finishes the proof of the lemma. □

B Phase diagrams for the Sobolev statistical solutions.

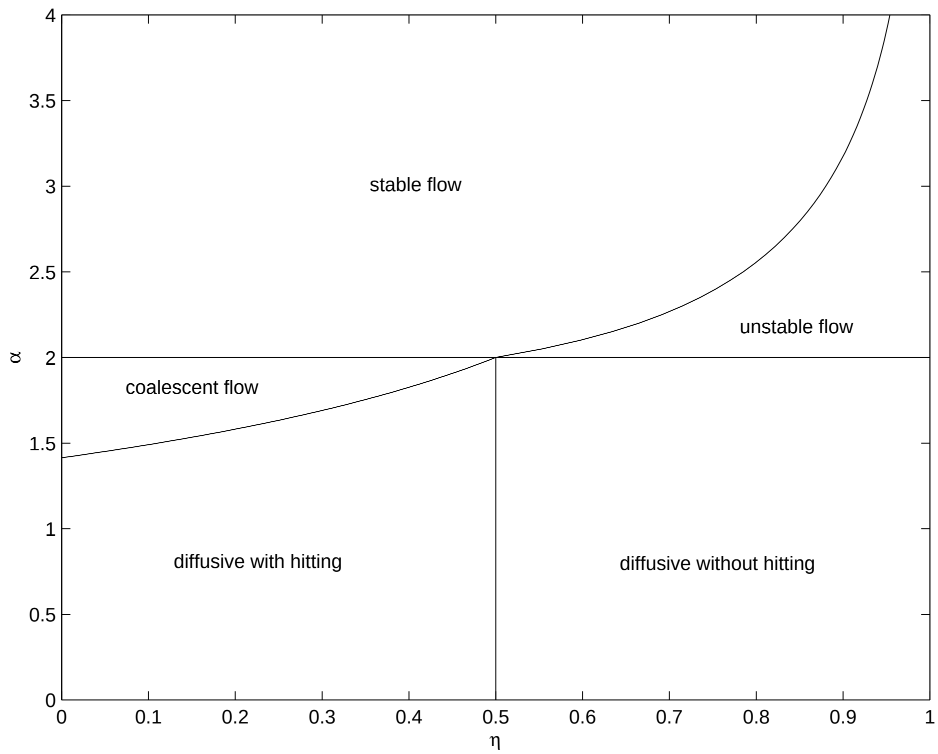

Figure 1 : Phase diagram on S^2.

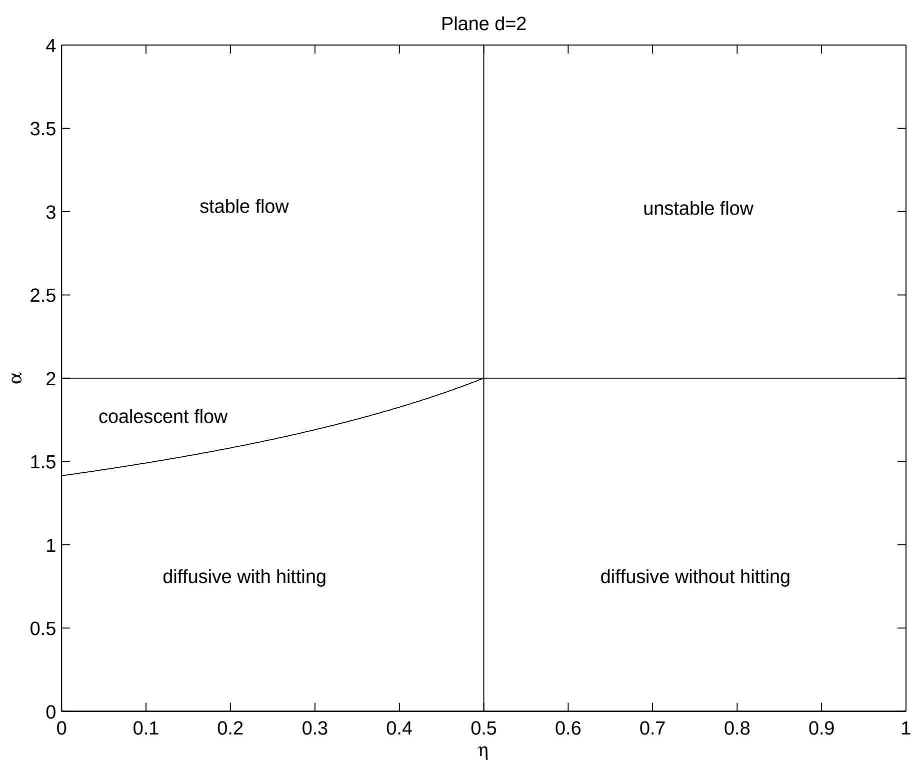

Figure 2 : Phase diagram on $\mathbb{R}^2$.

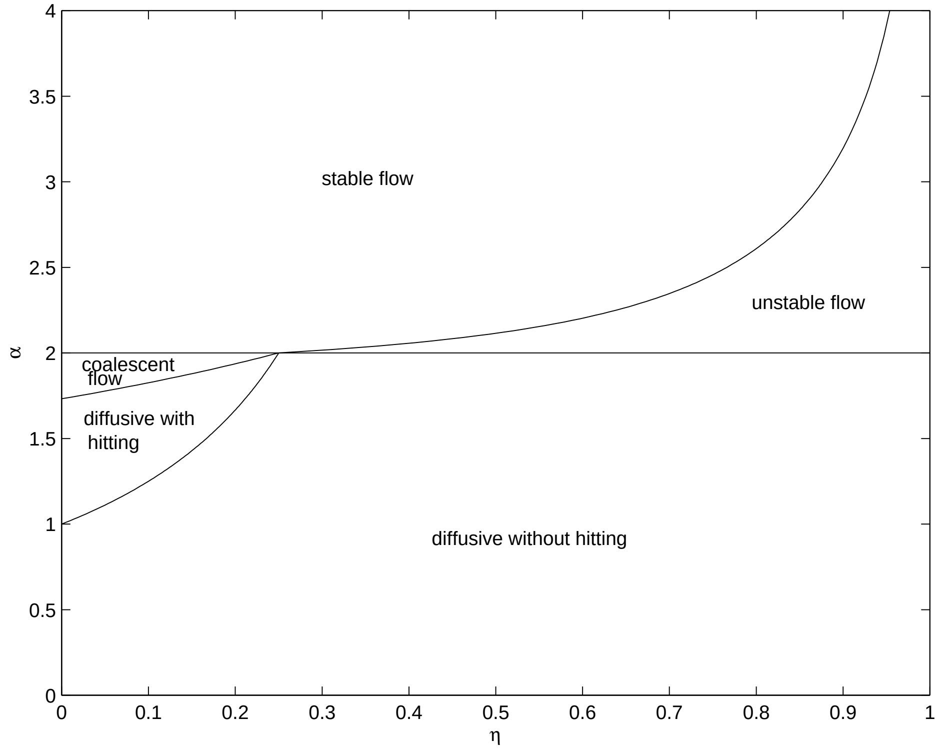

Figure 3 : Phase diagram on S^3.

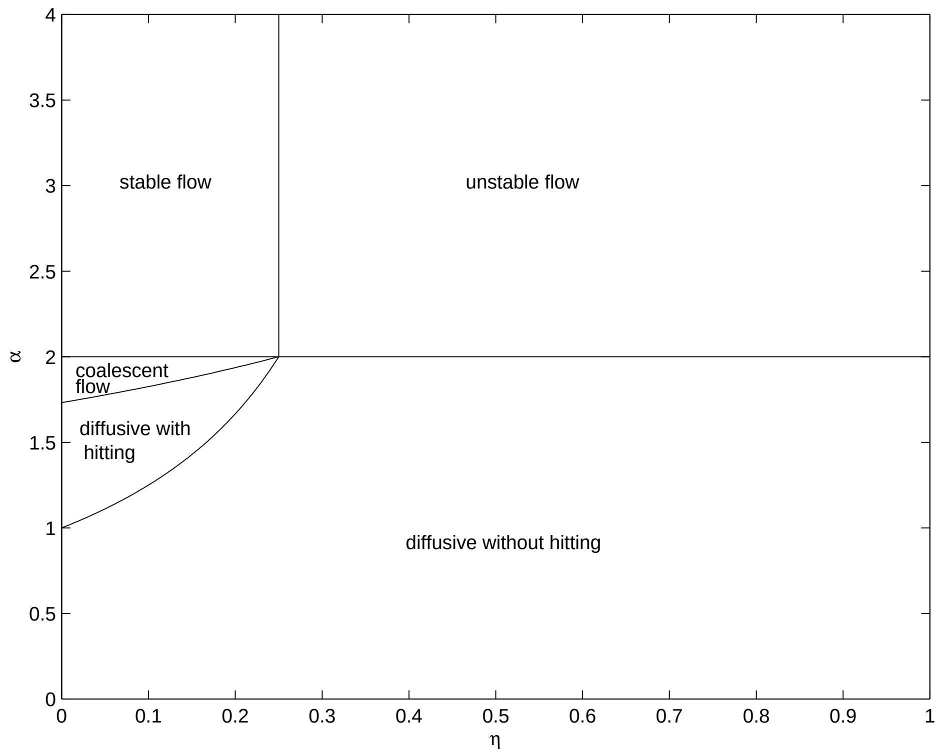

Figure 4 : Phase diagram on $\mathbb{R}^3$.

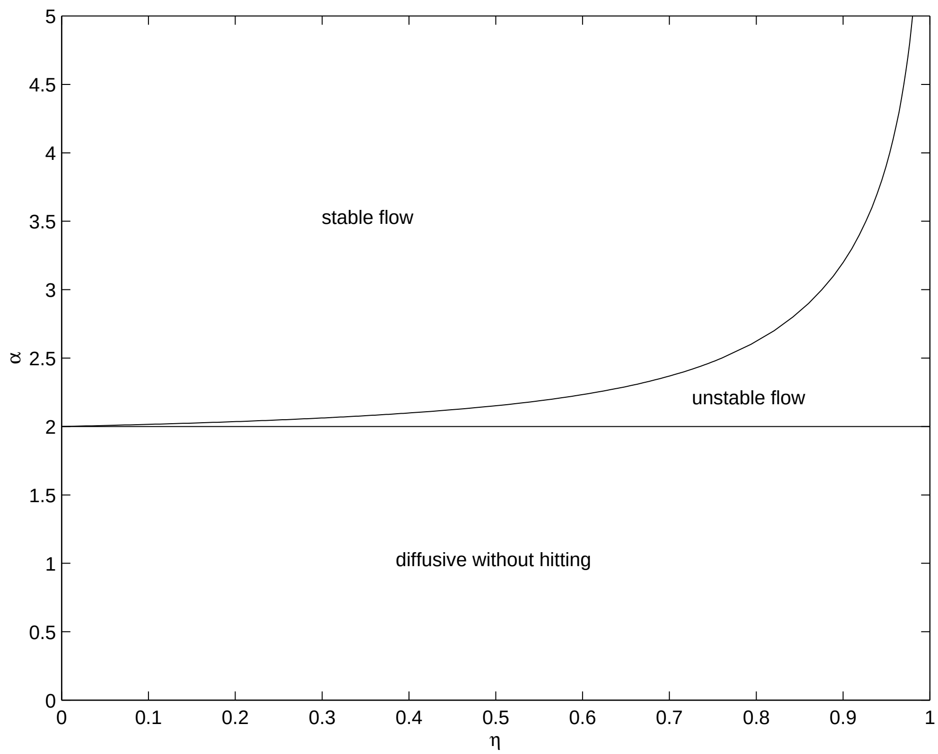

Figure 5 : Phase diagram on S^4.

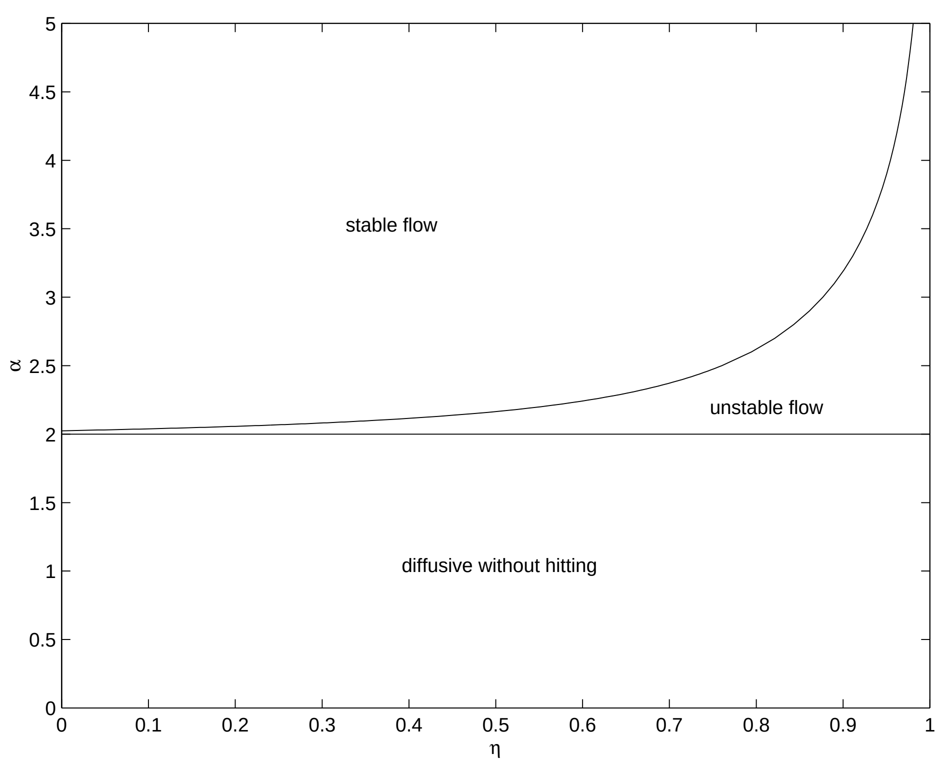

Figure 6 : Phase diagram on S^5.

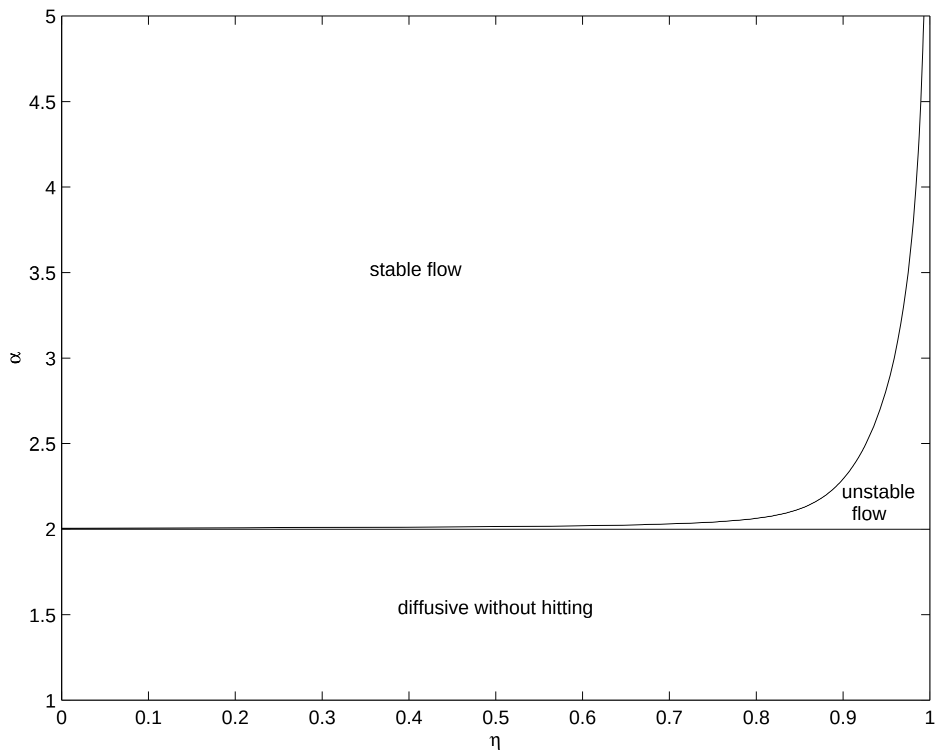

Figure 7 : Phase diagram on S^{50}.

Let us remark that when $\alpha < 2$, the diagrams are exactely the same for the sphere and for the plane. For the sphere, we see that, for $\alpha > 2$ and $\eta \leq 2 - \frac{\zeta(\alpha)}{\zeta(\alpha+1)}$, the flow gets stable when d goes to ∞ : (9.30) implies that $\lim_{d\to\infty} \eta(\alpha, d) = 2 - \frac{\zeta(\alpha)}{\zeta(\alpha+1)}$ for $\alpha > 2$. We see that, for any d and $\eta \in [0, 1[$, the flow gets stable when α goes to ∞ : (9.30) implies that $\lim_{\alpha\to\infty} \eta(\alpha, d) = 1$.

References

[1] BAXENDALE, P. (1976). Gaussian measures on function spaces. *American Journal of Mathematics*, **98**, No. 4, 891-952.

[2] BERNARD, D., GAWEDZKY, K. and KUPIAINEN, A. (1998). Slow modes in passive advection. *J. Stat. Phys.* **90**, 519-569.

[3] BOULEAU, N. and HIRSCH, F. (1991). Dirichlet forms and analysis on Wiener spaces. De Gruyter.

[4] BREIMAN, L. (1968). Probability. Addison Wesley.

[5] CHOQUET-BRUHAT, Y. and DEWITT-MORETTE, C. (1989). Analysis, manifolds and physics. North holland.

[6] CRANSTON, M. and LE JAN, Y. (1990). Noncoalescence for the Skorohod equation in a convex domain of $\mathbb{R}^d$. *Probab. Th. Rel. Fields* **87**, 241-252.

[7] DARLING, R.W.R. (1987). Constructing nonhomeomorphic stochastic flows. *Mem. Amer. Math. Soc.* **70**.

[8] DARLING, R.W.R. (1988). Rate of growth of the coalescent set in a coalescing stochastic flow. *Stochastics* **23**, 465-508.

[9] DARLING, R.W.R. and LE JAN, Y. (1988). The statistical equilibrium of an isotropic stochastic flow with negative Lyapunov exponent is trivial. *Séminaire de probabilités* **XXII**, 175-185.

[10] DE RHAM, G. (1955). Variétés différentiables. Formes, courants, formes harmoniques. Hermann.

[11] FUKUSHIMA, M., OSHIMA, Y. and TAKEDA, M. (1994). Dirichlet forms and symmetric Markov processes. de Gruyter.

[12] GALLOT, D. et MEYER, D. (1975). Opérateur de courbure et Laplacien des formes différentielles d'une variété riemannienne. *J. Math. pures et appl.* **54**, 259-284.

[13] GAWEDZKY, K. and KUPIAINEN, A. (1996). Universality in turbulence: an exactly solvable model. *Lecture notes in phys.* **469**.

[14] GAWEDZKY, K. and VERGASSOLA, M. (1998). Phase transition in the passive scalar advection. Preprint IHES/P/98/83.

[15] IKEDA, A. and TANIGUCHI, Y. (1978). Spectra and eigenforms of the Laplacian on S^n and $P^n(\mathbb{C})$. *Osaka J. Math.* **15**, 515-546.

[16] IKEDA, N. and WATANABE, S. (1988). Stochastic differential equations and diffusion processes. North Holland, 2nd ed.

[17] KRYLOV, N.V. and VERETENNIKOV, A.Ju. (1976). Explicit formulae for the solutions of the stochastic differential equations. *Math USSR Sb.* **29**, No. 2, 239-256.

[18] KUNITA, H. (1990) Stochastic flows and stochastic differential equations. Cambridge university press.

[19] KURTZ, T.G. and XIONG, J. (1999) Particle representations for a class of nonlinear S.P.D.E.'s. *Stochastic Process. Appl.* **83**, 103-126.

[20] LE JAN, Y. (1985). On Isotropic Brownian Motions. *Z. Wahrschein. verw. Gebiete* **70**, 609-620.

[21] LE JAN, Y. and RAIMOND, O. (1998). Solutions statistiques fortes des équations différentielles stochastiques. *C.R.A.S. Série I* **327**, 893-896.

[22] LE JAN, Y. and RAIMOND, O. math.PR/9909147.

[23] LE JAN, Y. and WATANABE, S. (1984). Stochastic flows of diffeomorphisms. *Proceedings of the Taniguchi Symposium 1982*. North-Holland.

[24] MALLIAVIN, P. (1997). Stochastic analysis. Springer.

[25] RAIMOND, O. (1999). Flots browniens isotropes sur la sphère. *Ann. Instit. H. Poincaré* **35**, 313-354.

[26] REVUZ, D. and YOR, M. (1999). Continuous martingales and Brownian motion. Springer, 3rd ed.

[27] VILENKIN, N.J. (1969). Fonctions spéciales et théorie de la représentation des groupes. Dunod.